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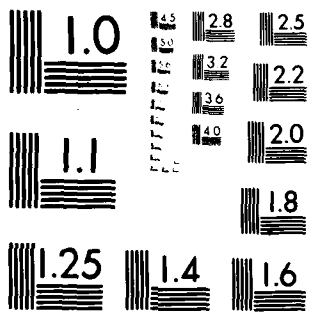

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Technical Report 569

OPTIMAL ESTIMATION OF MISSILE FREE-FLIGHT TRAJECTORY

Comparative results of linear and nonlinear Kalman filter approaches

F.D. Groutage,
Avionics Systems Branch,
Code 7311

Final Report: 10 July 1980

Prepared for:

Naval Air Systems Command,
Codes PMA 268 and 360H

Approved for public release; distribution unlimited

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ADMINISTRATIVE INFORMATION

Work was done in accordance with AIRTASK A360360H/003C/0W0981-006 by the Avionics Systems Branch, Code 7311, NOSC, under program element 63303N under the direction of the NAVAIR Electromagnetic Radiation Source Elimination (ERASE) Program Manager, Dennis Distler, AIR 360H. This work was conducted to support the Advanced Medium Range Air-to-Air Missile (AMRAAM) Program under the direction of NAVAIR PMA 268, Burton L. Munger, CAPT, USN.

Released by
FM Tirpak, Head
Sensors Division

Under authority of
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(14) NOSC/TR-341

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NOSC Technical Report 569 (TR 569)	2. GOVT ACCESSION NO. AD-A090 711	3. RECIPIENT'S CATALOG NUMBER (9)
4. TITLE (and Subtitle) OPTIMAL ESTIMATION OF MISSILE FREE-FLIGHT TRAJECTORY. Comparative results of linear and nonlinear Kalman filter approaches.		5. TYPE OF REPORT & PERIOD COVERED Final Report April 1980 – July 1980
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) F. Dale Groutage	W0???L	8. CONTRACT OR GRANT NUMBER(s) (17) W0981006
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Ocean Systems Center San Diego, CA 92152		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE63303N AIRTASK A360360H/003C/OW0981-006
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Air Systems Command Code 360H		12. REPORT DATE 10 July 1980 (12)
		13. NUMBER OF PAGES 62
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Estimation theory
Missile trajectory
Kalman filter
Extended Kalman filter

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is a technical report on the application of optimal estimation theory for estimating the trajectory of a free-flight missile, as a means of evaluating missile performance. Several approaches were investigated (linear Kalman filter working in Cartesian Coordinates and nonlinear extended Kalman filter implemented in Cartesian Coordinates). A six-degree of freedom simulation was used to generate the free-flight trajectory of the missile. A simulated radar, positioned at the origin of the inertial ground reference frame, generated range, azimuth, and elevation observation data. An analysis was conducted in the following three areas: (1) initialization, (2) Q matrix weighting, and (3) observation of noise variances. The nonlinear filter approach

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(linear dynamics with nonlinear measurements) proved to exhibit superior performance over a strictly linear filter approach. Further investigations, in specific areas, are recommended.

OBJECTIVE

Apply modern estimation theory to ground-based sensor tracking data for evaluating a missile's flight performance. The emphasis is on reconstruction of the missile's free-flight trajectory. The approaches investigated include the linear Kalman filter working in Cartesian coordinates, and the extended (nonlinear) Kalman filter implemented in Cartesian coordinates. A prime objective was an analysis of 1) filter initialization, 2) Q matrix weighting, and 3) variation in the statistics of the observation tracking sensor data.

RESULTS

The nonlinear filter approach (linear dynamics with nonlinear measurements) provided superior performance evaluation over a strictly linear filter approach. The use of an extended Kalman filter greatly enhances the level of confidence in defining (and reconstructing) a missile's free-flight trajectory.

RECOMMENDATIONS

It is recommended that a system engineering effort be conducted primarily in the areas of 1) adaptive Kalman filtering, including maneuver gates and weighted gain matrix; 2) initialization, including timing and initial values of covariance matrix, state vector, and variations in statistics of the noise on the sensor measurements; and (3) terminal intercept performance, including variance of line-of-sight rates measured by guidance sensors.

Secondary areas of investigation recommended are 1) nonlinear dynamics; 2) smoothing of optimal estimates; and 3) multiple sensor observations.

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1. ESTIMATION THEORY AS A TOOL

Modern estimation theory is one of the primary tools used to evaluate missile performance. Modern day missiles are tested in free-flight environments at well-instrumented test ranges. Sensors gather information about the missile as it is being tested in free flight. Estimation theory plays an important role in evaluating the observation information gathered by the range instrumentation sensors. The information is processed in a computerized data processing system. The data processing system performs a number of operations such as: (1) trajectory reconstruction of missile and target, (2) reconstruction of the missile attitudes and attitude rates throughout the flight, (3) determination of line-of-sight rates as seen by the missile sensor systems, (4) comparison of commanded versus achieved accelerations, (5) determination of the guidance and control steering commands which should compare to those generated by the missile flight control section, (6) comparison of measured line-of-sight rates as seen by the missile sensor system to the actual line-of-sight rates, (7) determination of the ability of the sensor to sort signals from noise and clutter, and (8) basic establishment of missile subsystem performance levels. This data analysis is utilized to establish the total missile system performance and to compare the free-flight performance against performance specifications.

ANALYSIS APPROACH

Reconstruction of the missile and target trajectories using modern day estimation theory, as emphasized in this report, is the key element in analyzing the missile's flight performance. Uncertainties in the measurements made by range sensors and system dynamics can be modeled and accounted for by using the mathematics of probability and statistics (ref 1). Figure 1 is a flow diagram of the flight data analysis process.

This report describes an analysis dealing primarily with optimal estimation theories, and mathematical tools for multivariate analysis (ref 2, 3, and 4) applied to reconstruction of missile free-flight trajectory. The approach is to generate a trajectory of a missile in free-flight using a six-degree of freedom simulation. Sections 3 and 6, and Appendix A present the details of this approach. The trajectory of the simulated missile flight

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1. Nahi, Nasser E., Estimation Theory and Applications, John Wiley and Sons, Inc., New York, N.Y., 1961.

2. Green, Paul E., Mathematical Tools for Applied Multivariate Analysis, Academic Press, Inc., 1978.

3. Chen, Chi-Tsong, Introduction to Linear Systems Theory; Holt, Rinehardt, and Winston, Inc., 1970.

4. Jazwinski, Andrew H., Stochastic Processes and Filtering Theory, Academic Press, New York, N.Y., 1970.

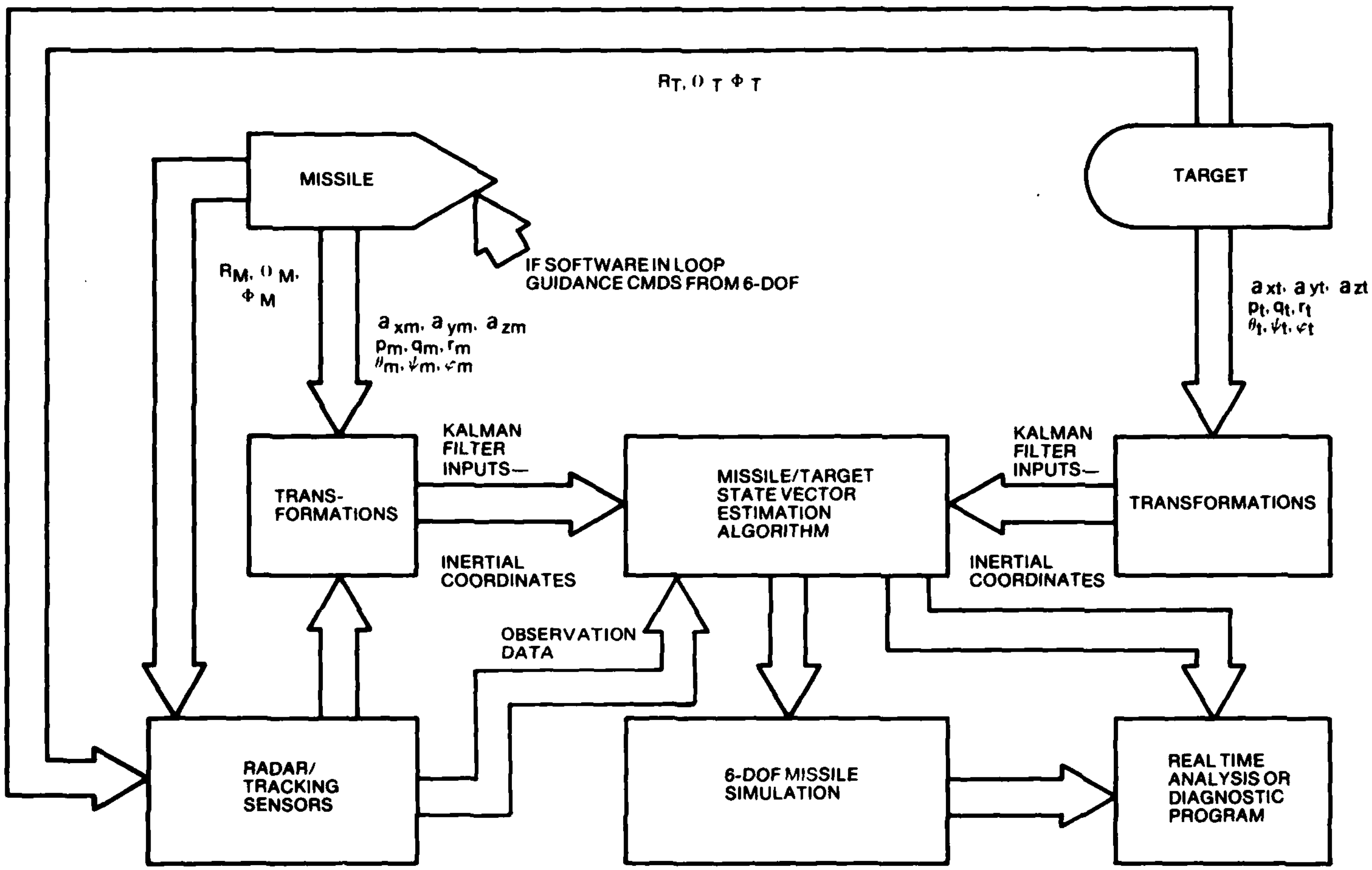

Figure 1. Flight data analysis process.

was observed using simulated ground-based sensors. The sensor data was processed through an optimal estimation sequential tracking filter.

Sections 2 and 3 define the reference frames, the models used in the study, and the formulation of the problem. Section 4 provides an overview of optimal estimation theory. Section 5 defines alternative sequential filter approaches for determining optimum estimates of the missile system state vector throughout the flight trajectory. A recommended approach is presented which is essentially an extended Kalman filter working in Cartesian coordinates.

Sections 5 and 6 are detailed descriptions of the filter that was modeled in a computer program and presents the analysis that was performed using this model. The analysis covered the following aspects of the problem: (1) initialization, (2) convergence, (3) mismodeled noise, (4) mismodeled system dynamics, (5) transformation noise, (6) data rates, and (7) advanced concepts relative to filter stability and rate of convergence.

Sections 6 and 7 present results, conclusions and recommendations.

2. SYSTEM REFERENCE FRAMES

This section defines the reference frames of the ground-based tracking sensors (defined for the analysis as the inertial frame) and the reference frames of the missile and of the target. The coordinate transformations that translate elements of the state vector from one frame to another are defined in Appendix B. The analysis for this study used one ground-based tracking sensor in the inertial coordinate system. In reality there would typically be a number of ground-based tracking sensors. The observation data of each individual tracking sensor would then be transformed to the inertial coordinate frame. The estimated values of the missile state vector for the missile throughout the flight trajectory would be the resultant of combining the observation data from all of the ground-based tracking sensors.

The inertial coordinate system is fixed to the surface of the earth with the positive Z axis directed towards the earth's center. The X_I axis is aligned parallel with the missile's initial launch direction. The ground-based tracking sensor (one was used for this analysis) is placed at the origin of the inertial frame. The missile reference system is fixed at the missile's center of gravity. The X_B axis coincides with the longitudinal axis while the Y_B and Z_B axes are directed along the pitch and yaw axes, respectively. The inertial and missile body reference frames are illustrated in figure 2.

The observation data obtained by ground-based sensor systems is range, range rate (if available), azimuth, and elevation angles. These variables are represented as R, $\dot{R}$, Θ and ϕ, respectively. It is immediately apparent that the observation data is a nonlinear function of the state variables as defined in a Cartesian coordinate system. For the purposes of this study the range rate, $\dot{R}$, was not utilized. The three remaining variables R, Θ and ϕ are illustrated in figure 3. The observation data is defined as

$$\underline{z}(t_k) = \underline{h}(\underline{x}_c(t_k)) + \underline{v}(t_k) \tag{1}$$

where $\underline{z}(t_k)$ is the P x 1 observation vector. Vectors in this report will be denoted as a lower case alphabetical letter underlined with a bar "$\underline{x}$."

In (1), $\underline{h}(\underline{x}_c(t_k))$ is the transformation function that relates R, Θ and ϕ, which are the position elements of the state vector in spherical coordinates ($\underline{x}_s$), to the three position elements x_c, y_c and z_c of the state vector in Cartesian coordinates. These functional relationships are defined in (2) through (7)

$$x_c = R\cos\Theta\cos\phi \tag{2}$$

$$y_c = R\sin\Theta\cos\phi \tag{3}$$

$$z_c = -R\sin\phi \tag{4}$$

$$R = (x_c{}^2 + y_c{}^2 + z_c{}^2)^{\frac{1}{2}} \tag{5}$$

$$\Theta = \arctan(y_c/x_c) \tag{6}$$

$$\phi = -\arctan[z_c(x_c{}^2 + y_c{}^2)^{-\frac{1}{2}}] \tag{7}$$

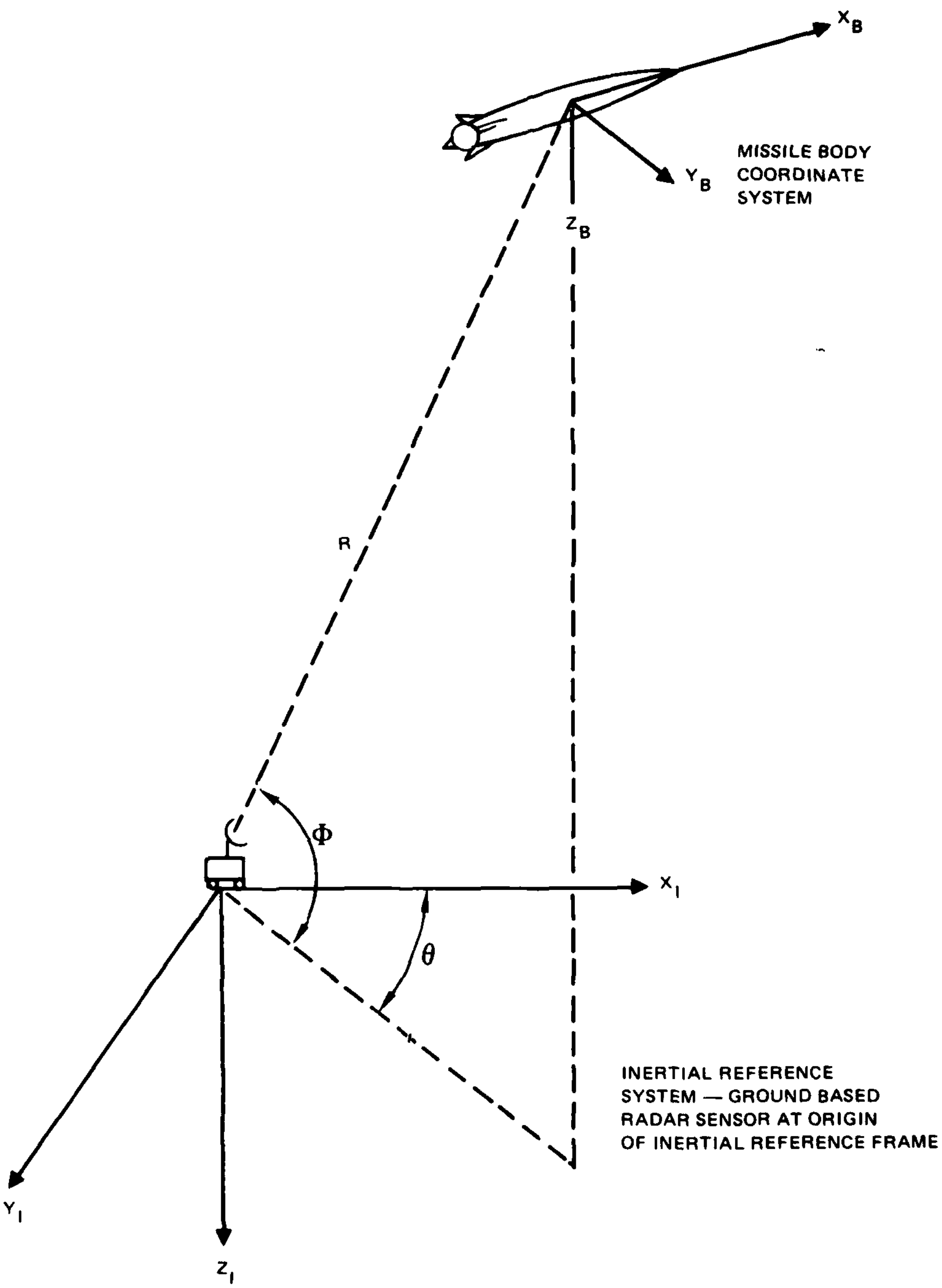

Figure 2. Missile body/inertial reference frames.

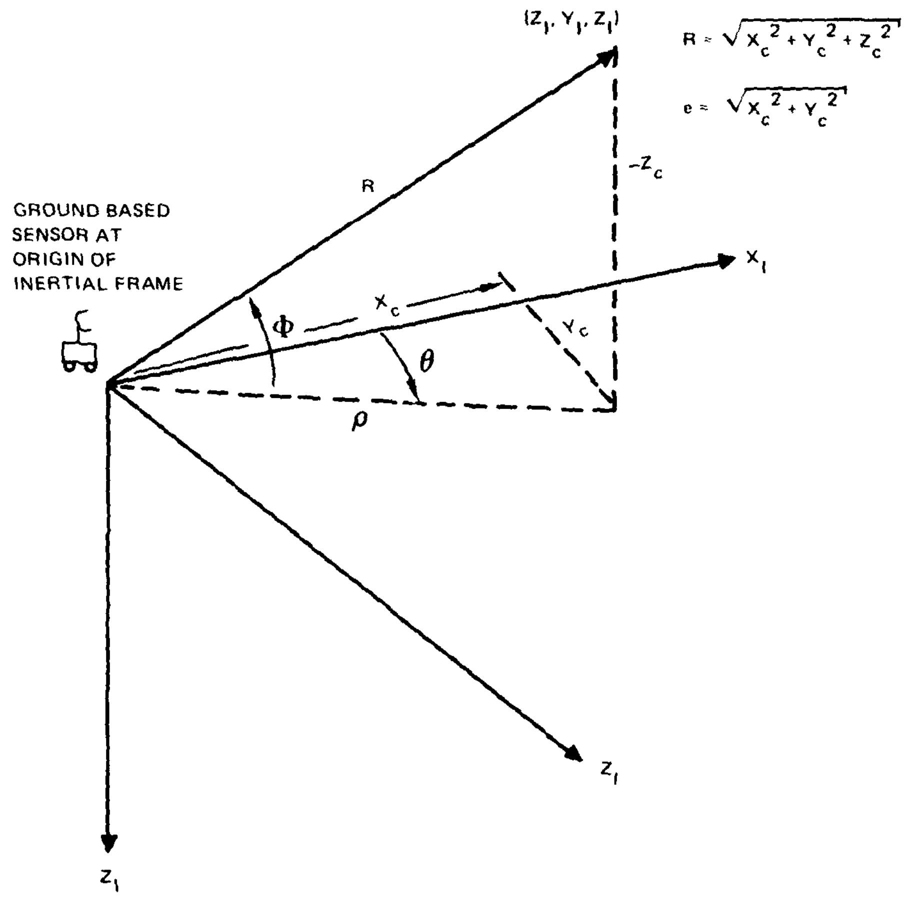

Figure 3. Spherical/rectangular coordinate systems.

3. MODELS AND PROBLEM FORMULATION

This section presents a brief description of the methodology, problem formulation, and the models that were used to generate the data for this study. Figure 4, an overview of the methodology, presents an expanded view of the anaysis and studies that would comprise a comprehensive missile trajectory estimation analysis effect. Only the shaded areas on the figure were covered in the current phase of the optimal estimation of missile free-flight performance study. The maneuver gate, Kalman gain weighting, guidance command comparison and estimation of line-of-sight rate during termir flight trajectory phase were not addressed in this study. These arc of study will be addressed in a follow-on study (as identified in figure 4).

The approach taken was to estimate a six-element state vector. The six-element state vector is composed of position and velocity elements and is written as follows:

$$\underline{x} = \begin{bmatrix} x_I \\ y_I \\ z_I \\ \dot{x}_I \\ \dot{y}_I \\ \dot{z}_I \end{bmatrix} \tag{8}$$

The inertial elements are in a Cartesian coordinate system.

Missile accelerations are assumed to be measurable quantities and would be available as deterministic forcing for the Kalman filter. These accelerations would be measured by missile on-board accelerometers. Similarly, the target's trajectory would be estimated by a six-element state vector where again it would be assumed that on-board target drone accelerometer information would be available for deterministic forcing.

Let's clarify what has been stated. This can be seen in terms of mathematical equations, and for brevity only the missile will be discussed. A parallel set of equations would describe the optimal estimation of the target drone trajectory. The dynamics of the system (missile system) in free flight can be defined in terms of the following nonlinear stochastic differential equation (ref 4 and 5):

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x},\underline{u},t) + \underline{w}(t),$$

$$\underline{x}(t_0) \sim N(\hat{\underline{x}}(t_0),\ P(t_0)) \tag{9}$$

where $\underline{x}$ is the n-vector of state variables describing the system, $\underline{u}$ is an n-vector of time-dependent forcing functions, $\underline{w}(t)$ is an n-vector of white

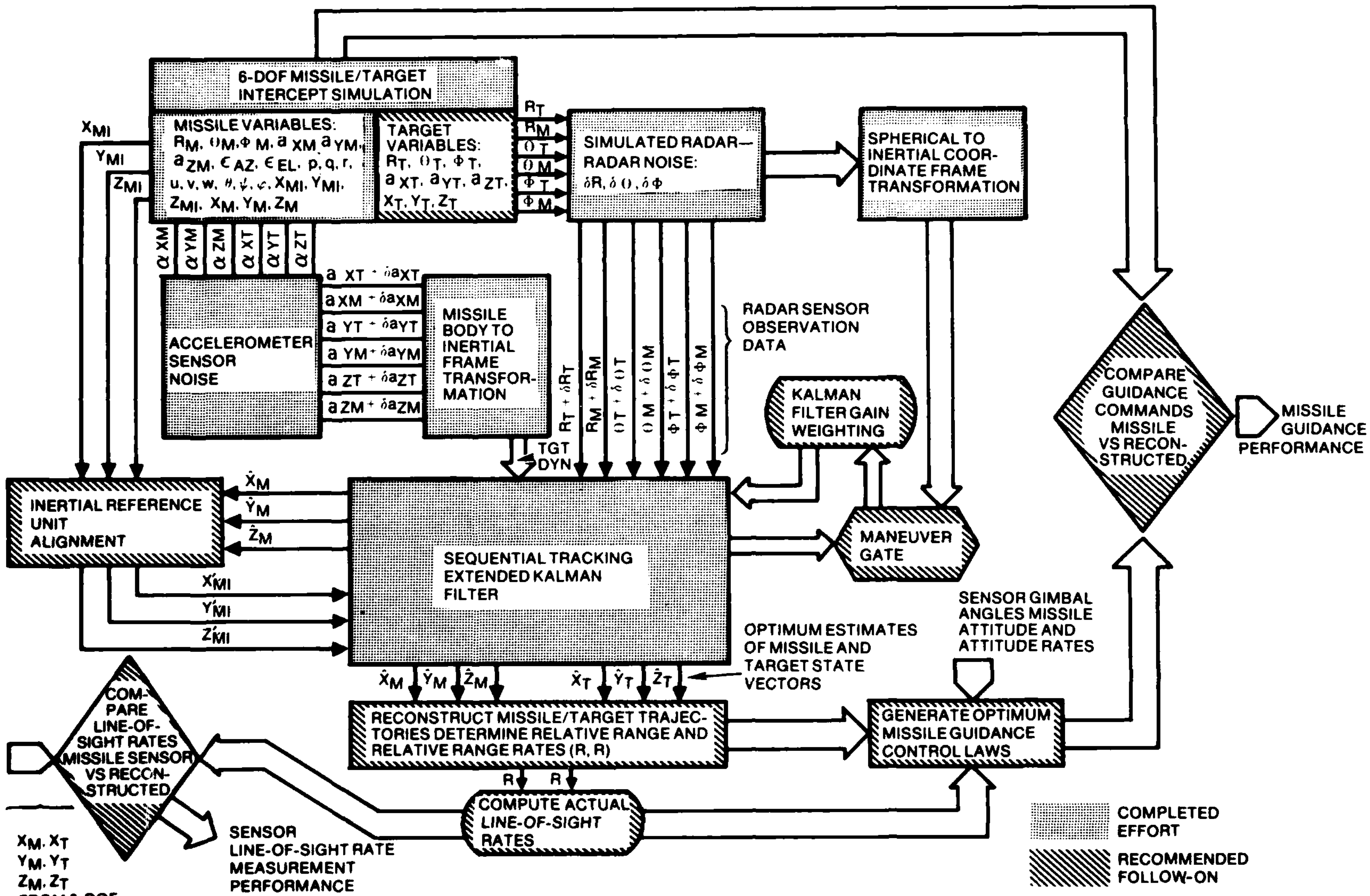

Figure 4. Methodology block diagram of missile performance evaluation from simulated free-flight test data.

process noise, t is the independent variable, time, and $\underline{f}$ is a known function of $\underline{x}$, $\underline{u}$ and t. The statistics on $\underline{w}(t)$ are

$$E\{\underline{w}(t)\} = 0 \tag{10}$$

and

$$E\{\underline{w}(t)\ \underline{w}^T(\tau)\} = Q(t)\delta(t-\tau) \tag{11}$$

where $Q(t)>0$ is the process noise spectral density matrix.

The discrete nonlinear observations[5] obtained by the ground-based sensor systems, which are taken at time instants t_k, can be expressed as

$$\underline{z}(t_k) = \underline{h}(\underline{x}(t_k)) + \underline{v}(t_k) \tag{12}$$

where $\underline{z}$ is the p-vector of observations of the state $\underline{x}$, $\underline{h}$ is the p-vector of nonlinear functions which relate the state and observations, and $\underline{v}$ is a p-vector of white measurement noise with statistics

$$E\{\underline{v}(t_k)\} = 0 \tag{13}$$

and

$$E\{\underline{v}(t_k)\underline{v}^T(t_k)\} = R(t_k) \tag{14}$$

The quantity $R(t_k)$ is the measurement noise covariance matrix which is restricted to be positive semidefinite.

Equation (9) describes the generalized system, including the nonlinear effects in the dynamics which relate to "real world" situations. One of the assumptions in this study is that the data rates for the accelerations are high enough that a linearized formulation of the system dynamics can be realized. In essence, the acceleration is assumed to be constant over an update interval (interval between data updates of the on-board accelerometers). It is also assumed that the system dynamics are time invariant. Based on these assumptions equation (9) is rewritten as

$$\dot{\underline{x}}(t) = F\underline{x}(t) + G\underline{u}(t) + \underline{w}(t) \tag{15}$$

where the solution of (16) is given as (reference 3 page 131)

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5. Applied Optimal Estimation, written by Technical Staff of the Analytic Sciences Corporation, edited by Arthur Gelb, The MIT Press, 1974.

$$\underline{x}(t) = \Phi(t, t_o)\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t,\tau)\, G\underline{u}(\tau)d\tau$$

$$+ \int_{t_o}^{t} \Phi(t,\tau)\, \underline{w}(\tau)d\tau \tag{16}$$

where $\Phi(t,\tau)$ is the state transition matrix of $\dot{\underline{x}}(t) = F\underline{x}(t)$; or, correspondingly, the unique solution of

$$\frac{\partial}{\partial t} \Phi(t,\tau) = F\Phi(t,\tau), \tag{17}$$

$$\Phi(\tau,\tau) = I$$

The F matrix is defined as

$$\begin{bmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix} \quad ; \tag{18}$$

which is the system matrix and the G matrix is defined as

$$\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \quad ; \tag{19}$$

which is the distribution matrix for the forcing function, missile acceleration, defined as

$$\underline{u}(t) = [a_x \; a_y \; a_z]^T \tag{20}$$

For the system model, the forcing function is the missile acceleration as measured by accelerometer sensors in the missile body axis system. The sensors measure with an error which is defined as the vector $\underline{w}(t)$, where

$$\underline{w}(t) = [0 \; 0 \; 0 \; \delta_{ax}(t) \; \delta_{ay}(t) \; \delta_{az}(t)]^T \tag{21}$$

The state transition matrix which is the solution of (17) can be solved using the inverse LaPlacian operator "L^{-1}"

$$\Phi(t,t_o) = L^{-1}[(SI - F)^{-1}] \tag{22}$$

For the system defined by (15) where F is given in (18), and when the system is discretely observed, the state transition matrix has the closed form solution

$$\Phi(\Delta) = \begin{bmatrix} 1 & 0 & 0 & \Delta & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta & 0 \\ 0 & 0 & 1 & 0 & 0 & \Delta \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

where Δ is the sampling interval for the discretely observed continuous system of (15).

We now have enough information to form an expression for the sampled version of (16):

$$\underline{x}(K + 1) = \Phi(\Delta)\underline{x}(K) + \int_{t_k}^{t_{k+1}} \Phi(\tau)G\underline{u}(\tau)d\tau \tag{24}$$

Equation (24) is termed as the update equation and where dealing with a stochastic system, the update is in terms of the optimum estimate of the state vector $\underline{x}$ at the Kth interval. Since the forcing function is assumed to be constant over the observation interval (assuming a high data rate/sampling rate on the missile acceleration) equation (24) can be written as

$$x(K+1) = \Phi(\Delta)\ \underline{x}(K) + \Gamma(\Delta)\ u(K) \tag{25}$$

where

$$\Gamma(\Delta) = \begin{bmatrix} \frac{\Delta^2}{2} & 0 & 0 \\ 0 & \frac{\Delta^2}{2} & 0 \\ 0 & 0 & \frac{\Delta^2}{2} \\ \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \tag{26}$$

At this stage the question arises, "why have we expended the efforts to define system dynamics as expressed in the previous development--equations (9) through (26)?" This question has two answers: First, the filter noted in figure (4) that will be utilized to formulate optimum estimates of the state vector will be a sequential Kalman filter. The estimates of $\underline{x}$ will be termed as $\hat{\underline{x}}$ where the "^" above the vector denotes an estimated value of the vector. A generalized model (ref 5, page 111) of a system model and a discrete Kalman filter is illustrated in figure 5.

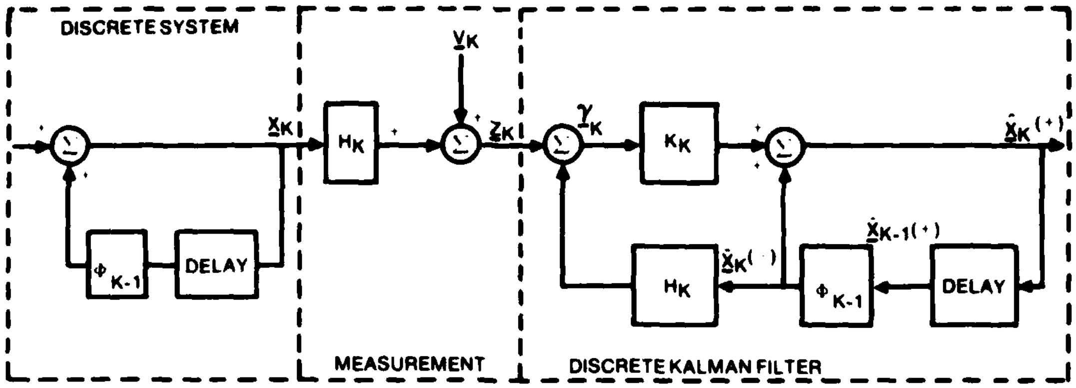

Figure 5. System model and discrete Kalman filter.

The Kalman filter illustrated in figure 5 requires a description of the system dynamics. Figure 5 does not represent the Kalman filter utilized in the tracking problem of figure 4, it merely represents a generalized Kalman filter to illustrate the point that the Kalman filter needs to have a knowledge of the system dynamics.

The second answer to the question relates to the fact that the analysis was performed on a digital computer and required a sampled data representation of the system. Having defined the system dynamics and a concept definition and approach (as illustrated in figure 4) the next step was to define the sequential filter that will derive optimum estimates of the state vector. This is a multi-faceted problem; therefore, we approach this through a sequence of logical tradeoffs and analysis. These steps are as follows:

1. Review optimal estimation theory. (This is an overview with reference made to the open literature for the details.)

2. Investigate alternate approaches; and

3. Analysis of recommended approaches.

These steps are detailed in the following three sections.

4. OPTIMAL ESTIMATION THEORY

This section presents a few fundamental ideas on optimal estimation theory. The emphasis is focused on nonlinear minimum variance estimation/extended Kalman filter theory.

An estimate, $\hat{\underline{x}}$ is the computed value of a quantity, $\underline{x}$, based on a set of measurements, $\underline{z}$. An unbiased estimate is one where expected value is the same as that of the quantity being estimated. A minimum variance/unbiased etimate has the property that its error variance is less than or equal to that of any other unbiased estimate. A consistent estimate is one which converges to the true value of $\underline{x}$, as the number of measurements increases. We shall then look at unbiased, minimum variance, consistent estimators.

The open literature (references 5, 6, 7, 8, 10) present details on various types of estimators. As an example ref 5 develops the theory for the least-squares and weighted-least-squares estimators. These are summarized as follows. The measurement process is modeled as

$$\underline{z} = H\underline{x} + \underline{v} \tag{27}$$

where $\underline{z}$ is an $\ell \times 1$ vector, $\underline{x}$ is on $n \times 1$ vector and H is an $\ell \times n$ matrix and $\underline{v}$ is an $\ell \times 1$ vector. The least squares estimator minimizes the quantity

$$J = (\underline{z} - H\hat{\underline{x}})^T (\underline{z} - H\hat{\underline{x}}) \tag{28}$$

The estimate is found by setting

$$\frac{\partial J}{\partial \hat{x}} = 0 \tag{29}$$

which results in the estimate

$$\hat{\underline{x}} = (H^T H)^{-1} H^T \underline{z} \tag{30}$$

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6. Frank J. Seiler Research Laboratory, SRL-TR-72-0004, An Engineer's Guide to Building Nonlinear Filters, Vol 1 and 2; Richard S. Bucy, Calvin Hecht, and Capt. Kenneth P. Semme, May 1972.

7. NELC TR 1967, Covariance Analysis of the DD963 Navigation System, by Jeffrey M. Nash, November 1975 (NELC is now NOSC).

8. MTR-2417, Understanding Kalman Filtering and Its Application in Real Time Tracking Systems, by J. J. Burke, the Mitre Corporation, Bedford, Mass., July 1972.

10. Mendel, Jerry M., Discrete Techniques of Parameter Estimation, Marcel Dekker, Inc., New York, N.Y., 1973.

The weighted-least-squares estimate is formed by minimizing

$$J = (\underline{z} - H\hat{\underline{x}})^T R^{-1} (\underline{z} - H\hat{\underline{x}}) \tag{31}$$

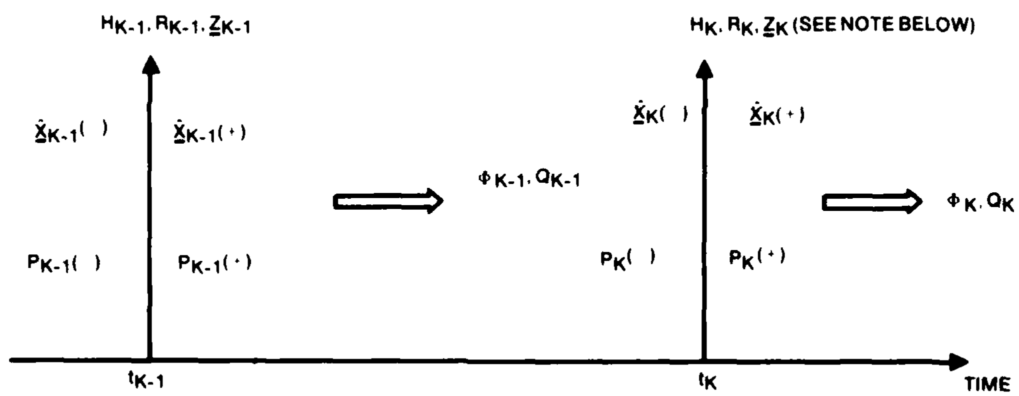
which results in the estimate

$$\hat{\underline{x}} = (H^T R^{-1} H)^{-1} H^T R^{-1} \underline{z} \tag{32}$$

The estimators described by equations (30) and (32) are referred to as Batch Processors. These estimators require that the observation data be stored. However, they are not sequential filters.

4.1 KALMAN FILTER EQUATIONS (LINEAR)

A recursive filter is one in which there is no need to store past measurements for the purpose of computing present estimates. This is basically the idea behind the Kalman filter. A timing diagram for a discrete Kalman filter is presented in figure 6. This diagram illustrates the propagation or extrapolation of both the state estimate $\hat{\underline{x}}_k(-)$ and the error covariance $P_k(-)$. The extrapolated values of the state estimate and error covariance are updated across each new measurement at discrete increments in time.



Note: the following convention for the notation will be assumed:

$\hat{\underline{x}}_{k-1}(-)$ is an equivalent form of $\hat{\underline{x}}(t_{k-1}^{-})$

$\hat{\underline{x}}_{k-1}(t)$ is an equivalent form of $\hat{\underline{x}}(t_{k-1}^{+})$

$P_{k-1}(-)$ is an equivalent form of $P(t_{k-1}^{-})$

etc.

Figure 6. Discrete Kalman filter timing diagram.

A summary of the discrete Kalman filter equation is outlined in table 1. The initial conditions and other assumptions are

$$E[\underline{x}(0)] = \hat{\underline{x}}(0),\ E[(\underline{x}(0) - \hat{\underline{x}}(0)(\underline{x}(0) - \hat{x}(0))^T] = P_0$$
$$E[\underline{w}_k V_j^T] = 0 \text{ for all } j,\ k \tag{33}$$

System Dynamics	$\dot{\underline{X}}(t) = F\,\underline{X}(t) + G\underline{U}(t) + \underline{W}(t),\ \underline{W}(t) \sim N(O,Q(t))$
State Transition Matrix	$\dot{\Phi}(t_K,t_{K-1}) = F\Phi(t_K,t_{K-1});\ \Phi(t_{K-1},t_{K-1}) = I$
Measurement Model	$\underline{Z}_K = H_K\underline{X}_K + \underline{V}_K,\ \underline{V}_K \sim N(O,R_K)$
State Estimate Extropolation	$\hat{\underline{X}}(t_K-) = \Phi(t_K,t_{K-1})\,\hat{\underline{X}}(t_{K-1}+) + \int_{t_{K-1}}^{t_K} \Phi(t_K,\tau)\,G(\tau)\,U(\tau)\,d\tau$
Error Covariance Extrapolation	$P(t_K-) = \Phi(t_K,t_{K-1})\,P(t_{K-1}+)\,\Phi^T(t_K,t_{K-1}) + \int_{t_{K-1}}^{t_K} \Phi(t_K,\tau)\,Q(\tau)\,\Phi^T(t_K,\tau)\,d\tau$
State Estimate Update	$\hat{\underline{X}}(t_K+) = \hat{\underline{X}}(t_K-) + K(t_K)\,[Z(t_K) - H(t_K)\,\hat{\underline{X}}(t_K-)]$
Error Covariance Update	$P(t_K+) = [I - K(t_K)\,H(t_K)]\,P(t_K-)$
Kalman Gain Matrix	$K(t_K) = P(t_K-)\,H^T(t_K)\,[H(t_K)\,P(t_K-)\,H^T(t_K) + R(t_K)]^{-1}$

Table 1. Kalman filter equations (linear dynamics and linear measurements).

4.2 NONLINEAR MINIMUM VARIANCE ESTIMATION (EXTENDED KALMAN FILTER)

The equations as outlined in table 1 are for a system described by linear dynamics and where the measurements are linear also. In reality, for most of the practical situations, neither the dynamics nor the measurements are linear. The Kalman filter is an algorithm where the conditional mean can be computed from a unique linear operation on the measurement data. For the more general case described by the nonlinear stochastic differential equation

$$\dot{x}(t) = \underline{f}(\underline{x}(t),t) + w(t) \tag{34}$$

and x(t) is estimated from sampled nonlinear measurements of the form

$$\underline{z}_k = \underline{h}_k(\underline{x}(t_k)) + \underline{v}_k \tag{35}$$

the problem of smoothing and filtering becomes considerably more difficult. For the linear gaussian case, the optimal estimate of $\underline{x}(t)$ for most reasonable Bayesian optimization criteria is the conditional mean. By contrast, in the nonlinear problem $\underline{x}(t)$ is generally not gaussian; hence many Bayesian criterion (ref 6) lead to estimates that are different from the conditional mean. More often, the nonlinear systems cannot be expressed in closed form, therefore methods of approximating optimal nonlinear filters must be devised. The remainder of this section will elaborate on the extended Kalman filter (one methodology for approximating optimal nonlinear filters). The covariance matrix is defined as

$$P(t) \triangleq E\left[[\underline{\hat{x}}(t) - \underline{x}(t)][\underline{\hat{x}}(t) - \underline{x}(t)]\right]^T \tag{36}$$

where the differential equation of P(t) is given as

$$\dot{P}(t) = E\Big[[\dot{\underline{\hat{x}}}(t) - \dot{\underline{x}}(t)][\underline{\hat{x}}(t) - \underline{x}(t)]^T + [\hat{x}(t) - \underline{x}(t)][\dot{\underline{\hat{x}}}(t) - \dot{\underline{x}}(t)]^T\Big] \tag{37}$$

$$\dot{\hat{x}}(t) = \hat{\underline{f}}(\underline{x}(t),t) \tag{38}$$

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t),t) + \underline{w}(t) \tag{39}$$

Working only with the first terms of equation (37) and where the dependence of $\underline{x}$ on t and $\underline{f}$ and on $\underline{x}$ and t is suppressed for notational convenience we have

$$\begin{aligned} &E[(\hat{\underline{f}} - \underline{f} - \underline{w})(\underline{\hat{x}} - \underline{x})^T] \\ &= E[\hat{\underline{f}}\ \underline{\hat{x}}^T - \hat{\underline{f}}\ \underline{x}^T - \underline{f}\ \underline{\hat{x}}^T + \underline{f}\ \underline{x}^T \\ &\quad - \underline{w}\underline{\hat{x}}^T + \underline{w}\underline{x}^T] \qquad (40) \\ &= [\hat{\underline{f}}\underline{\hat{x}}^T - \hat{\underline{f}}\underline{\hat{x}}^T - \hat{\underline{f}}\underline{\hat{x}}^T + \widehat{\underline{f}\underline{x}}^T \\ &\quad - \hat{\underline{w}}\underline{\hat{x}}T + \widehat{\underline{w}\underline{x}}^T \end{aligned}$$

$$= - \hat{\underline{f}}\hat{\underline{x}}^T + \widehat{\underline{f}\underline{x}}^T + \widehat{\underline{w}\underline{x}}^T \tag{41}$$

remembering that

$$\underline{x} = \underline{x}(t_{k-1}) + \int_{t_{k-1}}^{t} \underline{f}(\underline{x}(\tau),\tau)d\tau + \int_{t_{k-1}}^{t} \underline{w}(\tau)d\tau \tag{42}$$

$$E[\underline{w}\underline{x}^T] = E[\underline{w}(x^T(t_{k-1}) + \int_{t_{k-1}}^{t} \underline{f}^T(\underline{x}(\tau),\tau)d\tau + \int_{t_{k-1}}^{t} \underline{w}^T(\tau)d\tau)] \tag{43}$$

where $E[\underline{w}] = 0$

Thus the first two terms of (43) drop out and the only term left is

$$E[\underline{w}(t) \int_{t_{k-1}}^{t} \underline{w}^T(\tau)d\tau] \tag{44}$$

Interchanging the integration and expectation operation and evaluating we get

$$\widehat{\underline{w}\underline{x}}^T = 1/2Q \tag{45}$$

where

$$E[\underline{w}(t)\underline{w}^T(\tau)] = Q(t)\,\delta\,(t-\tau) \tag{46}$$

Looking now at the second terms of equation (37):

$$E\left[[\hat{\underline{x}}(t) - \underline{x}(t)]\,[\dot{\hat{\underline{x}}}(t) - \dot{\underline{x}}(t)]^T\right] \tag{47}$$

and again making the substitutions for $\dot{\hat{x}}$ and $\dot{x}$ into (47) and using simplified notation as mentioned above we get

$$E[\hat{\underline{x}}\dot{\hat{\underline{x}}}^T - \hat{\underline{x}}\dot{\underline{x}}^T - \underline{x}\dot{\hat{\underline{x}}}^T + \underline{x}\dot{\underline{x}}^T] \tag{48}$$

$$= E[\hat{\underline{x}}\hat{\underline{f}}^T - \hat{\underline{x}}(\underline{f}^T + \underline{w}^T) - \underline{x}\hat{\underline{f}}^T + \underline{x}(\underline{f}^T + \underline{w}^T)]$$

$$= \underline{\hat{x}}\hat{\underline{f}}^T - \underline{\hat{x}}\hat{\underline{f}}^T - \underline{\hat{x}}\underline{\hat{w}}^T - \underline{\hat{x}}\hat{\underline{f}}^T + E[\underline{x}\underline{f}^T] + E[\underline{x}\underline{w}^T]$$

$$= - \underline{\hat{x}}\hat{\underline{f}}^T + \widehat{\underline{x}\underline{f}}^T + 1/2Q \qquad (49)$$

Combining the results of (41) and (49) we get the composite or

$$\dot{P} = \widehat{\underline{f}\underline{x}}^T - \hat{\underline{f}}\underline{\hat{x}}^T - \underline{\hat{x}}\hat{\underline{f}}^T + \widehat{\underline{x}\underline{f}}^T + Q \qquad (50)$$

Equation (50) is the desired result, and inspection of (50) shows that $\dot{P}$ is a function of $\underline{f}$ which depends on the probability density function p(x,t) or

$$\hat{\underline{f}}(\underline{x},t) = \int_{-\infty}^{\infty} \underline{f}(\underline{x},t)\, p(\underline{x},t)dx_1 . . . d_{xn} \qquad (51)$$

Thus, to compute $\hat{\underline{f}}(\underline{x},t)$, the probability density p(x,t) must be known. An alternative, when full information is not known about the probability function p(x,t), is to linearize the system dynamics. One methodology is to use a Taylor series expansion of $\underline{f}(x,t)$ and truncate the series after several terms. This will be referred to as Quasi Linearization.

Define

$$\dot{\underline{x}}(t) = \underline{f}(\underline{\hat{x}},t) + F(\underline{x} - \underline{\hat{x}}) + \underline{w}(t) \qquad (52)$$

$$\dot{\underline{\hat{x}}}(t) = \hat{\underline{f}}(\underline{x},t) = \underline{f}(\underline{\hat{x}},t) \qquad (53)$$

$$\dot{P}(t) = E\Big[\Big(\underline{f}(\underline{\hat{x}},t) - \underline{f}(\hat{x},t) - F(\underline{x} - \underline{\hat{x}}) - \underline{w}(t)\Big)\Big(\underline{\hat{x}}(t) - \underline{x}(t)\Big)^T\Big] + E\Big[\Big(\underline{\hat{x}}(t) - \underline{x}(t)\Big)\Big(\underline{f}(\underline{\hat{x}},t) - \underline{f}(\underline{\hat{x}},t) - F(\underline{x} - \underline{\hat{x}}) - \underline{w}(t)\Big)^T\Big]$$

$$= E\Big[\Big(-F(\underline{x} - \underline{\hat{x}}) - \underline{w}(t)\Big)\Big(\underline{\hat{x}}(t) - \underline{x}(t)\Big)^T\Big] + E\Big[\Big(\underline{\hat{x}}(t) - \underline{x}(t)\Big)\Big(-F(\underline{x} - \underline{\hat{x}}) - \underline{w}(t)\Big)^T\Big] \qquad (54)$$

$$\dot{P}(t) = FP + 1/2Q + PF^T + 1/2Q \qquad (55)$$

where $F = \left. \frac{\partial \underline{f}(\underline{x},t)}{\partial \underline{x}} \right|_{\underline{x} = \hat{\underline{x}}}$

and

$$\dot{\hat{\underline{x}}}(t) = \underline{f}(\hat{\underline{x}},t) \tag{56}$$

Equations (54) and (55) are approximate expressions for propagating the conditional mean of the state and its associated covariance matrix. (For more details see ref 4.) These are the propagation equations of the filter algorithm. To complete the algorithm update equations are required which weight the observation data. The Kalman filter algorithm formulates an update as a linear function of the measurement

$$\hat{\underline{x}}(t_k+) = \underline{a}(t_k) + K(t_k)\ \underline{z}(t_k) \tag{57}$$

The estimation errors are defined as

$$\underline{\varepsilon}(t_k+) \triangleq \hat{\underline{x}}(t_k+) - \underline{x}(t_k) \tag{58}$$

$$\underline{\varepsilon}(t_k-) \triangleq \hat{\underline{x}}(t_k-) - \underline{x}(t_k) \tag{59}$$

Combining (57), (58) and (59) yields

$$\underline{\varepsilon}(t_k+) = \underline{a}(t_k) + K(t_k)\ \underline{h}(\underline{x}(t_k)) + K(t_k)\ \underline{v}(t_k) + \underline{\varepsilon}(t_k-) - \hat{\underline{x}}(t_k-) \tag{60}$$

Utilizing the fact that the estimate is unbiased, ie., that $E[\underline{\varepsilon}(t_k+)] = 0$ and that

$$E[\underline{\varepsilon}(t_k-)] = 0 \tag{61}$$

and $E[\underline{v}(t_k)] = 0$

an expression for $a(t_k)$ can be solved for using (60). This expression for $a(t_k)$, is formulated by taking the expectation of (60) and applying the conditions of (61). This resultant expression for $a(t_k)$,

$$\underline{a}(t_k) = \hat{\underline{x}}(t_k-) - K(t_k)\ \hat{\underline{h}}\big(\underline{x}(t_k)\big) \tag{62}$$

is substituted into (57) which yields an expression for the update estimate

$$\hat{\underline{x}}(t_k+) = \hat{\underline{x}}(t_k-) + K(t_k)\left[\underline{z}(t_k) - \hat{\underline{h}}\left(\underline{x}(t_k)\right)\right] \tag{63}$$

An expression for the estimation error can also be formulated using (60) and (62)

$$\underline{\varepsilon}(t_k+) = \underline{\varepsilon}(t_k-) + K(t_k)\left[\underline{h}\left(\underline{x}(t_k)\right) - \hat{\underline{h}}\left(\underline{x}(t_k)\right)\right] + K(t_k)\,\underline{v}(t_k) \tag{64}$$

The optimal gain is required which will yield an estimate (the approximate conditional mean of x(t)) which is a minimum variance estimate--one that minimizes the class of functions

$$J(t_k) = E[\underline{\varepsilon}(t_k+)^T\, S\, \underline{\varepsilon}(t_k+)] \tag{65}$$

for any positive semidefinite matrix S. By choosing S to be the identity matrix I, then

$$J(t_k) = E[\underline{\varepsilon}(t_k+)^T\, \underline{\varepsilon}(t_k+)] = \text{trace}\,[P(t_k+)] \tag{66}$$

where

$$P(t_k+) = E[\underline{\varepsilon}(t_k+)\, \underline{\varepsilon}(t_k+)^T] \tag{67}$$

By expanding (67) out and taking the trace and solving the equation

$$\frac{\partial J(t_k)}{\partial K(t_k)} = 0 \tag{68}$$

for $K(t_k)$ yields

$$\begin{aligned} K(t_k) = -E\left[\underline{\varepsilon}(t_k-)\left[\underline{h}\left(\underline{x}(t_k)\right) - \hat{\underline{h}}\left(\underline{x}(t_k)\right)\right]^T\right. \\ \times \left\{E\left[\left[\underline{h}\left(\underline{x}(t_k)\right) - \hat{\underline{h}}\left(\underline{x}(t_k)\right)\right]\left[\underline{h}\left(\underline{x}(t_k)\right)\right.\right.\right. \\ \left.\left.\left. - \underline{h}\left(\underline{x}(t_k)\right)\right]^T + R(t_k)\right\}^{-1} \end{aligned} \tag{69}$$

and

$$P(t_k+) = P(t_k-) + K(t_k)E\left[\left[\underline{h}(\underline{x}(t_k)) - \hat{\underline{h}}\left(\left(\underline{x}(t_k)\right)\right]\underline{\varepsilon}(t_k-)^T\right] \tag{70}$$

Equations (69) and (70) are impractical to implement because of their dependence upon the probability density function for $\underline{x}(t)$ which is required to calculate $\underline{h}(t_k)$. A solution would be to expand $\underline{h}(x(t_k))$ in a Taylor series about $\hat{\underline{x}}(t_k)$.

$$\underline{h}\left(\underline{x}(t_k)\right) = \underline{h}\left(\hat{\underline{x}}(t_k-)\right) + H\left(\underline{x}(t_k-)\right)\left(\underline{x}(t_k) - \hat{\underline{x}}(t_k)\right) + \ldots \tag{71}$$

where

$$H\left(\underline{x}(t_k-)\right) = \left.\frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}}\right|_{\underline{X} = \hat{\underline{x}}(t_k-)} \tag{72}$$

Truncating the series of equation (71) after two terms and substituting the approximation for $\underline{h}(x(t_k))$ into equations (69) and (70) and the equation

$$\hat{\underline{h}}\left(x(t_k)\right) = \underline{h}\left(\hat{\underline{x}}(t_k)\right) \tag{73}$$

which is obtained from the approximation of (71) results in the extended Kalman filter update equation as follows:

$$\hat{\underline{x}}(t_k+) = \hat{\underline{x}}(t_k-) + K(t_k)\left[\underline{z}(t_k) - \underline{h}\left(\hat{\underline{x}}(t_k-)\right)\right] \tag{74}$$

$$K(t_k) = P(t_k-)\, H^T\left(\hat{\underline{x}}(t_k-)\right)\left[H\left(\hat{\underline{x}}(t_k-)\right)P(t_k-)\, H^T\left(\hat{\underline{x}}(t_k-)\right) + R(t_k)\right]^{-1} \tag{75}$$

$$P(t_k+) = \left[I - K(t_k)\, H\left(\hat{\underline{x}}(t_k-)\right)\right]P(t_k-) \tag{76}$$

Equations (53), (55), (74), (75), and (76) constitute the extended Kalman filter equations. These equations are summarized in table 2.

System Dynamics	$\dot{\underline{X}}(t) = \underline{f}(\underline{X}(t),t) + \underline{W}(t);\ \underline{W}(t) \sim N(O,Q(t))$
Measurement Model	$\underline{Z}(t_K) = \underline{h}(\underline{X}(t_K)) + \underline{V}(t_K)$
Initial Conditions Assumption	$\underline{X}(O) \sim N(\hat{\underline{X}}_o, P_o)$ $E[\underline{W}(t)\underline{V}(t_K)^T] = O$ for all t and all k
State Estimate Propagation	$\dot{\hat{X}}(t) = \underline{f}(\hat{\underline{X}}(t), t)$
Error Covariance Propagation	$\dot{P}(t) = F(\hat{\underline{X}}(t,t)\, P(t) + P(t)\, F^T(\hat{\underline{X}}(t),t) + Q(t)$
State Estimate Update Error Covariance Update Gain Matrix	$\hat{\underline{X}}_K(+) = \hat{\underline{X}}_K(-) + K_K\,[\underline{Z}_K - \underline{h}_K(\hat{\underline{X}}_K(-))]$ $P_K(+) = [I - K_K\, H_K(\hat{\underline{X}}_K(-))]\, P_K(-)$ $K_K = P_K(-)\, H_K{}^T(\hat{\underline{X}}_K(-))\left[H_K(\hat{\underline{X}}_K(-))\, P_K(-) \cdot H_K{}^T(\hat{X}_K(-)) + R_K\right]^{-1}$
Definitions	$F(\hat{\underline{X}}(t),t) = \left.\frac{\partial f(\underline{X}(t),t)}{\partial \underline{X}(t)}\right\vert_{\underline{X}(t) = \hat{\underline{X}}(t)}$ $H_K(\hat{\underline{X}}(-)) = \left.\frac{\partial \underline{h}_K(\underline{X}(t_K))}{\partial \underline{X}(t_K)}\right\vert_{\underline{X}(t_K) = \hat{\underline{X}}(-)}$

Table 2. Extended Kalman filter equations.

5. ALTERNATE APPROACHES

This section describes the alternate approaches to formulating the Kalman filter for estimating the state variables which define the trajectory of a missile during free flight. Two types of filters were investigated for the tracking algorithm: (1) strictly linear (linear dynamics and linear measurements) in Cartesian coordinates, and (2) nonlinear (linear dynamics but nonlinear measurements) implemented in Cartesian coordinates. A nonlinear filter implemented in spherical coordinates was also investigated.

5.1 LINEAR FILTER APPROACH

The first approach (strictly linear filter in Cartesian coordinates) utilized a noise variance transformation methodology to allow the filter observation to be linear. The state vector is defined as

$$x(t) = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ z \\ \dot{z} \end{bmatrix} \tag{77}$$

and the state transition matrix defined as

$$\Phi = \begin{bmatrix} 1 & \Delta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \Delta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \Delta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{78}$$

Burke (ref 8) outlines a methodology for decoupling the filter. That is, have three parallel filters with state transition matricies of order 2 x 2 rather than one filter with a 6 x 6 state transition matrix. There is some loss of information by going to the decoupled filters--some of the cross terms in the covariance matrix are lost. Since all three filters of the decoupled arrangement are essentially identical, only one will be elaborated upon. The x component will be used as the example to define the algorithm. Analogous y and z filter are similarly derived. The state vector (for x element) is now a 2 x 1 vector.

$$\underline{x}(t) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{79}$$

where the state transition matrix is

$$\Phi = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \tag{80}$$

Δ is the sampling time for the discrete system dynamics.

$$\underline{x}(t_k) = \Phi(t_k, k-1)\ \underline{\hat{x}}(t_k-1) + \int_{t_{k-1}}^{t_k} \Phi(t_k,\tau)G\underline{u}(\tau)d\tau \tag{81}$$

where

$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{82}$$

and

$$\underline{u}(t) = \underline{a}_x(t) + \underline{\delta}_a \tag{83}$$

$a_x(t)$ is the missile lateral acceleration in the ground inertial frame and δ_{ax} is the noise component.

The translation accelerations in the missile body system are

$$\zeta_1 = (\dot{u} + Qw + Rv - G_x) \tag{84}$$

$$\zeta_2 = (\dot{v} + R_u - Pw - G_y)$$

$$\zeta_3 = (\dot{w} + P_v - Q_u - G_z)$$

The quantities $\dot{u}$, $\dot{v}$, $\dot{w}$, and P, Q and R are the rectilinear and angular accelerations of the missile (missile body coordinates) as defined in ref (9). G_x, G_y, and G_z are components of gravity in x, y, and z directions, respectively.

The translational acceleration components in the inertial frame are

$$\begin{aligned} a_x &= \zeta_1 A_{11} + \zeta_2 a_{21} + \zeta_3 a_{31} \\ a_y &= \zeta_1 A_{12} + \zeta_2 a_{22} + \zeta_3 a_{32} \\ a_z &= \zeta_1 A_{13} + \zeta_2 a_{23} + \zeta_3 a_{33} \end{aligned} \tag{85}$$

the A_{ij} terms are defined in Appendix A. The $\underline{\delta}_a$ term in (83) is the nondeterministic portion of the forcing function.

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9. McRuer, Duane; Ashkemas, Irving; and Graham, Dustan, Aircraft Dynamics and Automatic Control, Princeton University Press.

The equations (3) through (7) are the functional relationships that define the observation data in spherical with respect to the Cartesian coordinate frame or in the Cartesian frame with respect to the elements in the spherical frame. These equations are used as the basis for the noise variance transformation from spherical to the Cartesian frame.

The differentials of equations (3) through (5) are:

$$\delta x_c = \delta R\cos\theta\cos\phi - R\delta\theta\sin\theta\cos\phi - R\delta\phi\cos\theta\sin\phi$$
$$\delta y_c = \delta R\sin\theta\cos\phi + R\delta\theta\cos\theta\cos\phi - R\delta\phi\sin\theta\sin\phi \qquad (86)$$
$$\delta z_c = -\delta R\sin\phi - R\delta\phi\cos\phi$$

The noise variance can now be defined as:

$$E[\delta x_c{}^2] = \sigma_R{}^2\cos^2\theta\cos^2\phi + R^2\sigma_\theta{}^2\sin\theta^2\cos^2\phi + \sigma_\phi{}^2R^2\cos^2\theta\sin^2\phi$$
$$E[\delta y_c{}^2] = \sigma_R{}^2\sin^2\theta\cos^2\phi + R^2\sigma_\theta{}^2\cos^2\theta\cos^2\phi + R^2\sigma_\phi{}^2\sin^2\theta\sin^2\phi \qquad (87)$$
$$E[\delta z_c{}^2] = \sigma_R{}^2\sin^2\phi + R^2\sigma_\phi{}^2\cos^2\phi$$

where

$$\sigma_R{}^2 = E[\delta R^2]$$
$$\sigma_\theta{}^2 = E[\delta\theta^2] \qquad (88)$$
$$\sigma_\phi{}^2 = E[\delta\phi^2]$$

The linear Kalman filter will be completely defined once the initial conditions are defined. The initial conditions needed to start the filter are $\hat{x}_o$, P_o, and Q_o.

The initial estimate of the state vector is determined as follows. The position element of the state vector will be the second usable measurement of position from the sensor. Hence, a two point start is assumed. The velocity component is approximated using Euler's definition of the derivative from the first and second measurements of position. The initial values of the elements of the covariance matrix are calculated as follows; with covariance matrix defined as

$$P = E[\delta\underline{x}\ \delta\underline{x}^T) \qquad (89)$$

where

$$\delta\underline{x} = \begin{bmatrix} \delta x_c \\ \delta\dot{x}_c \end{bmatrix} \qquad (90)$$

$$P = E\begin{bmatrix} \delta x_c\delta x_c & \delta x_c\delta\dot{x}_c \\ \delta\dot{x}_c\delta x_c & \delta\dot{x}_c\delta\dot{x}_c \end{bmatrix} \qquad (91)$$

δx_c is defined in equation (86) using the definition for the Euler integration

$$\delta\dot{x}_c(t_k) \cong \frac{\delta x_c(t_k) - \delta x_c(t_{k-1})}{T} \tag{92}$$

$$E[\delta\dot{x}_c(t_k)^2] = \frac{2\sigma_{\delta x}{}^2}{\Delta^2} \tag{93}$$

where

$$\sigma_{\delta x}{}^2 = E[\delta x\ \delta x] \tag{94}$$

Using similar logic

$$E[\delta x_c(t_k)\ \delta\dot{x}_c(t_k)] = \frac{\sigma_{\delta x}{}^2}{\Delta} \tag{95}$$

Thus

$$P_o = \begin{bmatrix} \sigma_{\delta x_c}{}^2 & \dfrac{\sigma_{\delta xc}{}^2}{\Delta} \\ \dfrac{\sigma_{\delta x_c}{}^2}{\Delta} & \dfrac{2\sigma_{\delta xc}{}^2}{\Delta^2} \end{bmatrix} \tag{96}$$

This completes derivation of the decoupled linear (Cartesian coordinate) filter algorithm. The equations for this filter algorithm are those of table 1 where the following initial condition and assumptions were made:

1. Linear dynamics with state transition matrix

$$\Phi = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}$$

2. Forcing function distribution matrix

$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3. Forcing function is missile translational accelerations measured by onboard accelerometers

4. $\underline{w}(t)$ is accelerometer sensor noise variance

5. Ground tracking radar sensor noise variances in Cartesian coordinates are defined in equation (92)

6. Initial value of $\underline{x}_0$ is derived from the first two consecutive useable measurements of the observation data and initial value of P, P_0 is defined in equation (96)

5.2 EXTENDED KALMAN FILTER APPROACH

The second approach, which is the recommended approach and implemented as a computerized algorithm, is the nonlinear extended Kalman filter (linear dynamics but nonlinear measurements) implemented in Cartesian coordinates. The state vector $\underline{x}(t)$, of the dynamic system (missile) in flight is defined as

$$\underline{x}(t) = \begin{bmatrix} x_c \\ y_c \\ z_c \\ \dot{x}_c \\ \dot{y}_c \\ \dot{z}_c \end{bmatrix} \tag{97}$$

and the state evolves as the following:

$$\dot{\underline{x}}(t) = F\underline{x}(t) + B\underline{u}(t) + \underline{w}(t) \tag{98}$$

where

$\underline{x}(t)$ is the n x 1 vector of system state variables,

$\underline{u}(t)$ is the P x 1 vector of deterministic forcing functions,

$\underline{w}(t)$ is the q x 1 vector of the random forcing functions,

F is the n x n system matrix,

G is the n x P distribution matrix for deterministic forcing.

The solution to (98), which is used to propagate the state vector, is

$$\hat{\underline{x}}(t_k^-) = \Phi(t_k, t_{k-1})\hat{\underline{x}}(t_{k-1}^+) + \int_{t_{k-1}}^{t_k} \Phi(t_{k-1},\tau)G\underline{u}(\tau)d\tau$$

$$+ \int_{t_{k-1}}^{t_k} \Phi((t_k, \tau)\underline{w}(\tau)d\tau \tag{99}$$

where

$$\frac{\partial\Phi(t,\tau)}{\partial t} = F\Phi(t,\tau),\ \Phi(\tau,\tau) = I \tag{100}$$

and

$$\Phi(t,t_o) = L^{-1}[(SI - F)^{-1}] \tag{101}$$

and the system matrix, F, is

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{102}$$

The solution to the differential equation defined in (55) which propagates the covariance matrix is (see ref 5)

$$P(t_k-) = \Phi(t_k,\ t_{k-1})\ P(t_{k-1}+)\Phi^T(t_k,\ t_{k-1}) + \int_{t_{k-1}}^{t_k} \Phi(t_k,\ \tau)Q(\tau)\Phi^T(t_k,\ \tau)d\tau \tag{103}$$

The G matrix is defined as

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{104}$$

which is the distribution matrix for forcing function, missile acceleration, defined as

$$\underline{u}(t) = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \tag{105}$$

and

$$\underline{w}(t) \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta a_x \\ \delta a_y \\ \delta a_z \end{bmatrix} \tag{106}$$

which is the random forcing of the system.

The variance of the random forcing Q(t) is defined as

$$Q(t)\delta(t - \tau) = E\,\{\underline{w}(t)\underline{w}^T(\tau)\} \tag{107}$$

$$Q = E \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta a_x^{\,2} & \delta a_x \delta a_y & \delta a_x \delta a_z \\ 0 & 0 & 0 & \delta a_y \delta a_x & \delta a_y^{\,2} & \delta a_y \delta a_z \\ 0 & 0 & 0 & \delta a_z \delta a_x & \delta a_z \delta a_y & \delta a_z^{\,2} \end{bmatrix} \tag{108}$$

The measurement equation is given as

$$\underline{z}(t_k) = \underline{h}\big(\underline{x}(t_k)\big) + \underline{v}(t_k) \tag{109}$$

with the partials, $H(\hat{\underline{x}}(t_k-)$, defined as

$$H\big(\hat{\underline{x}}(t_k-)\big) = \frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}}\bigg|_{\underline{x} = \hat{\underline{x}}(t_k-)} \tag{110}$$

and restating the functional relationships,

$\underline{h}\big(\underline{x}(t_k)\big)$, as

$$\underline{h}\big(\underline{x}(t_k)\big) \triangleq \begin{cases} R(t_k) = (x_c^{\,2}(t_k) + y_c^{\,2}(t_k) + z_c^{\,2}(t_k))^{1/2} \\ \Theta(t_k) = \mathrm{ATAN}\big(y_c(t_k)/x_c(t_k)\big) \\ \phi(t_k) = -\mathrm{ATAN}\Big[z_c(t_k)\big(x_c^{\,2}(t_k) + y_c(t_k)\big)^{-1/2}\Big] \end{cases} \tag{111}$$

The partials which comprise the elements of the measurement matrix are:

$$H(1,1) = \frac{\partial R(x,y,z)}{\partial x}\bigg|_{\underline{x} = \hat{x}_k(-)} = \hat{x}_k(-)\Big(\hat{x}_k^{\,2}(-) + \hat{y}_k^{\,2}(-) + \hat{z}_k^{\,2}(-)\Big)^{-1/2}$$

$$H(1,2) = \frac{\partial R(x,y,z)}{\partial y}\bigg|_{\underline{x} = \hat{x}_k(-)} = \hat{y}_k(-)\Big(\hat{x}_k^{\,2}(-) + \hat{y}_k^{\,2}(-) + \hat{z}_k^{\,2}(-)\Big)^{-1/2}$$

$$H(1,3) = \frac{\partial R(x,y,z)}{\partial z}\bigg|_{\underline{x} = \hat{x}_k(-)} = \hat{z}_k(-)\Big(\hat{x}_k^{\,2}(-) + \hat{y}_k^{\,2}(-) + \hat{z}_k^{\,2}(-)\Big)^{-1/2}$$

$$H(1,4) = \frac{\partial R(x,y,z)}{\partial \dot{x}}\bigg|_{\underline{x} = \hat{x}_k(-)} = 0 \tag{112}$$

$$\partial(1,5) = \left.\frac{\partial R(x,y,z)}{\partial \dot{y}}\right|_{\underline{x} = \hat{x}_k(-)} = 0$$

$$\partial(1,6) = \left.\frac{\partial R(x,y,z)}{\partial \dot{z}}\right|_{\underline{x} = \hat{x}_k(-)} = 0$$

$$H(2,1) = \left.\frac{\partial \theta(x,y)}{\partial x}\right|_{\underline{x} = \hat{x}_k} = \frac{-\hat{y}_k(-)}{\hat{x}_k{}^2(-) + \hat{y}_k{}^2(-)}$$

$$H(2,2) = \left.\frac{\partial \theta(x,y)}{\partial y}\right|_{\underline{x} = \hat{x}_k} = \frac{\hat{x}_k(-)}{\hat{x}_k{}^2(-) + \hat{y}_k{}^2(-)}$$

$$H(2,3) = \frac{\partial \theta(x,y)}{\partial z} = 0$$

$$H(2,4) = \frac{\partial \theta(x,y)}{\partial \dot{x}} = 0 \qquad (112)\ \text{continued}$$

$$H(2,5) = \frac{\partial \theta(x,y)}{\partial \dot{y}} = 0$$

$$H(2,6) = \frac{\partial \theta(x,y)}{\partial \dot{z}} = 0$$

$$H(3,1) = \left.\frac{\partial \phi(x,y,z)}{\partial x}\right|_{\underline{x}_k = \underline{\hat{x}}_k} = \frac{\hat{x}_k(-)\hat{z}_k(-)}{\left(\hat{x}_k{}^2(-) + \hat{y}_k{}^2(-) + \hat{z}_k{}^2(-)\right)\left(\hat{x}_k{}^2(-) + \hat{y}_k{}^2(-)\right)^{1/2}}$$

$$H(3,2) = \left.\frac{\partial \phi(x,y,z)}{\partial y}\right|_{\underline{x}_k = \underline{\hat{x}}_k} = \frac{\hat{y}_k(-)\hat{z}_k(-)}{\left(\hat{x}_k{}^2(-) + \hat{y}_k{}^2(-) + \hat{z}_k{}^2(-)\right)\left(\hat{x}_k{}^2(-) + \hat{y}_k{}^2(-)\right)^{1/2}}$$

$$H(3,3) = \left.\frac{\partial \phi(x,y,z)}{\partial z}\right|_{\underline{x}_k = \underline{\hat{x}}_k} = \frac{\left(-\hat{x}_k(-) + \hat{y}_k(-)\right)^{1/2}}{\left(\hat{x}_k{}^2(-) + \hat{y}_k{}^2(-) + \hat{z}_k{}^2(-)\right)}$$

$$H(3,4) = \frac{\partial \phi(x,y,z)}{\partial \dot{x}} = 0$$

(112) continued

$$H(3,5) = \frac{\partial\phi(x,y,z)}{\partial\dot{y}} = 0$$

$$H(3,6) = \frac{\partial\phi(x,y,z)}{\partial\dot{z}} = 0$$

The update equations for the extended Kalman filter are:

$$K(t_k) = P(t_k-)H^T\left(\hat{\underline{x}}(t_k-)\right)\left[H\left(\hat{\underline{x}}(t_k-)\right)P(t_k-)H^T\left(\underline{x}(t_k-)\right) + R(t_k)\right]^{-1} \quad (113)$$

$$\hat{\underline{x}}(t_k+) = \hat{\underline{x}}(t_k-) + K(t_k)\left[\underline{z}(t_k) - \underline{h}\left(\hat{\underline{x}}(t_k-)\right)\right] \quad (114)$$

$$P(t_k+) = \left[I - K(t_k)\, H\left(\hat{\underline{x}}(t_k-)\right)\right]P(t_k-) \quad (115)$$

The filter was initialized using the same procedure as defined for the strictly linear Kalman filter described earlier in this section. The initial covariance matrix, however, was a 6 x 6 matrix rather than a 2 x 2. The initial covariance matrix was

$$P_o = E\begin{bmatrix} \delta x_c\delta x_c & \delta x_c\delta y_c & \delta x_c\delta z_c & \delta x_c\delta\dot{x}_c & \delta x_c\delta\dot{y}_c & \delta x_c\delta\dot{z}_c \\ \delta y_c\delta x_c & \delta y_c\delta y_c & \delta y_c\delta z_c & \delta y_c\delta\dot{x}_c & \delta y_c\delta\dot{y}_c & \delta y_c\delta\dot{z}_c \\ \delta z_c\delta x_c & \delta z_c\delta y_c & \delta z_c\delta z_c & \delta z_c\delta\dot{x}_c & \delta z_c\delta\dot{y}_c & \delta z_c\delta\dot{z}_c \\ \delta\dot{x}_c\delta x_c & \delta\dot{x}_c\delta y_c & \delta\dot{x}_c\delta z_c & \delta\dot{x}_c\delta\dot{x}_c & \delta\dot{x}_c\delta\dot{y}_c & \delta\dot{x}_c\delta\dot{z}_c \\ \delta\dot{y}_c\delta x_c & \delta\dot{y}_c\delta y_c & \delta\dot{y}_c\delta z_c & \delta\dot{y}_c\delta\dot{x}_c & \delta\dot{y}_c\delta\dot{y}_c & \delta\dot{y}_c\delta\dot{z}_c \\ \delta\dot{z}_c\delta x_c & \delta\dot{z}_c\delta y_c & \delta\dot{z}_c\delta z_c & \delta\dot{z}_c\delta\dot{x}_c & \delta\dot{z}_c\delta\dot{y}_c & \delta\dot{z}_c\delta\dot{z}_c \end{bmatrix} \quad (116)$$

In the above equation δx, δy, and δz are defined as per equation (86). $\delta\dot{x}$, $\delta\dot{y}$, and $\delta\dot{z}$ are defined as in the Euler approximations to the derivative. The evaluation of each term is performed by first multiplying the appropriate term and then taking the expectation of the products. It is easily seen that the algebra gets very involved, thus an approximation for obtaining P_o was investigated for the analysis in this study.

6. ANALYSIS OF RECOMMENDED APPROACH AND RESULTS

The recommended approach for deriving estimates of the state vector which defines the trajectory of the missile and or target in free-flight is the extended Kalman filter as described in the previous section. This approach was studied using the methodology of figure 4, i.e., a six degree of freedom simulation was used to simulate the missile in free flight. Missile translational acceleration data as defined by equations (85) comprises the forcing function. A ground-based simulated radar generated tracking or observation data as the missile pursues the target. A series of analyses were conducted using a baseline missile/target intercept geometry. Filter performance was investigated as a function of: (1) the initial condition x_o, and P_o, (2) weighting the covariance matrix, and sensor noise statistics.

6.1 BASELINE MISSILE AND TARGET INTERCEPT

The initial values for the missile and target baseline flight simulator are outlined in table 3.

Missile:	
x_m	0.0 feet
y_m	0.0 feet
z_m	60,000.0 feet
Speed at Launch	Mach 0.9
Target:	
x_T	84,000.0 feet
y_T	30,573.0 feet
z_T	60,000.0 feet
Seed (Constant Velocity)	Mach 0.8

Table 3. Flight simulation baseline values.

A diagram of the XY plane of the missile target intercept geometry is illustrated in figure 7. Since the missile and target were at a co-altitude for the launch, the XY plane is the plane of interest. This baseline trajectory was used for all the filter performance analysis that was conducted.

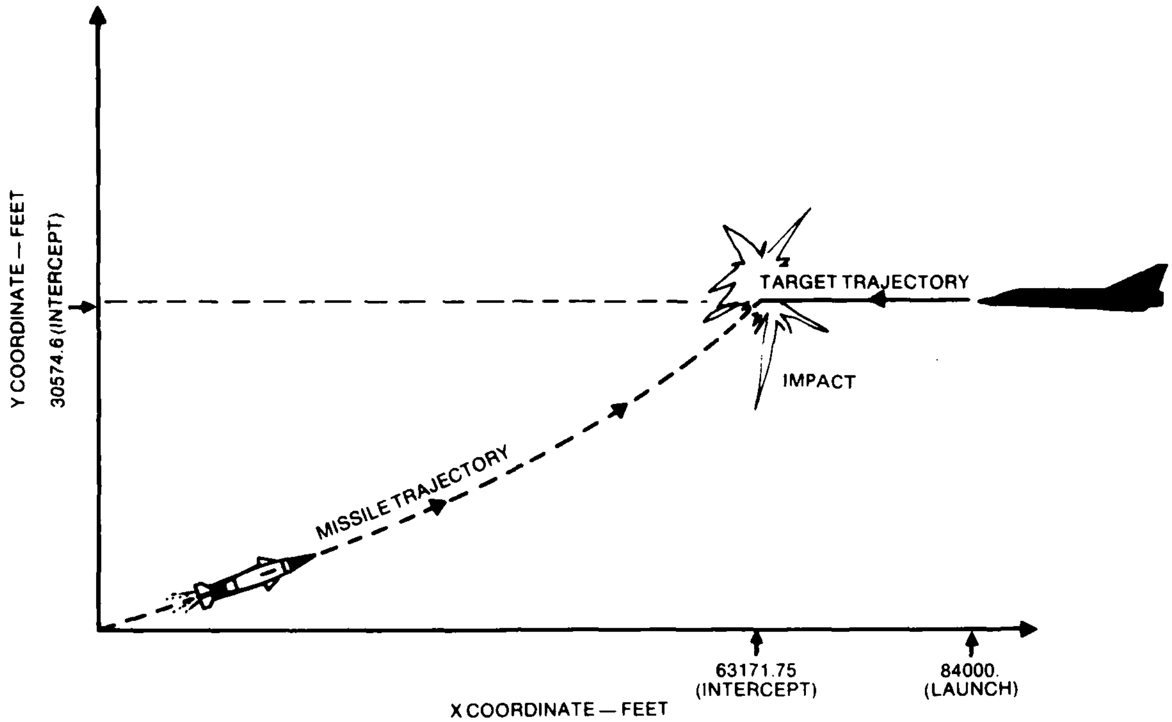

Figure 7. Missile/target intercept geometry. (XY plane of trajectory)

6.2 FILTER PERFORMANCE ANALYSIS

Figures 8 through 16 present performance analysis data for the extended Kalman filter. Figures 17 through 19 are comparative data of the linear filter. Two sets of statistical performance parameters (variance on range, azimuth, and elevation) were used for the analysis. Ref (11) defines the noise standard deviations for two radars at the White Sands Missile Range. The standard deviations (sigmas, "σ") are as follows:

	σ_R	σ_A	σ_E
FPS-16's	5 ft	.1 mil	.1 mil
MPS-36's	5 ft	.2 mil	.2 mil

The radar statistical error data used in this analysis was defined as a low error case and a large error case. Even the low error case was of a higher noise level than that of ref 11. This was because the objective of the

11. Mathematical Service Branch, U.S. Army White Sands Missile Range, Technical Report No. 58, Optimal Radar Instrumentation Planning, by William S. Agee, White Sands Missile Range, New Mexico.

analysis was to thoroughly evaluate the filter's capabilities operating with poor observation data--the theory being that if the filter can obtain reasonable estimates with very noisy observations, it will yield better estimates with less noisy observation data. The statistics defining the two individual cases are as follows:

	σ_R	σ_A	σ_E
Low Noise (case 1)	10 ft	.316 mil Rad	.316 mil Rad

	σ_R	σ_A	σ_E
High Noise (case 2)	31.62 ft	3.16 mil Rad	3.16 mil Rad

6.2.1 Extended Kalman Filter Analysis

Figures 8 through 11 present data for Case 1 (x component). Figure 8 presents two plots: (1) error of actual raw data, and (2) error in the estimated value of the x component of the missile flight trajectory. The curves clearly illustrate that a large reduction in the measurement error is obtained by filtering. Figure 9 presents four data plots. These are plots of the RMS errors $z - \hat{x}$, $z - x$, and $x - \hat{x}$. The method of computing the standard deviation of sampled data was that of ref 12, i.e.,

$$\sigma_\Theta = \left[\sum_{i=1}^{N} \frac{(x_i)^2}{N-1} \right]^{1/2} \tag{117}$$

In figure 9, two curves for the RMS error of $x - \hat{x}$ are plotted. One curve starts the RMS formula of equation (11) at time equal to 1 second; neglecting the data from time zero up to 1 second. The second curve starts the RMS formula at time equal to 6 seconds, i.e., $i = 1$ at $t = 6$. The first 6 seconds of data are not weighted in the RMS calculation. It was noted that the convergence time of the filter (with the diagonal elements of the Q matrix being 1×10^4) was roughly 6 seconds. Figure 9 clearly indicates the level of performance of the extended Kalman filter. The estimated values of the x component of the trajectory are four to six times better than the raw data measurements of the x component of the missile trajectory. This is based on the ratio of RMS errors.

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12. Bendat, Julius, and Piersol, Allan G., Random Data Analysis and Measurement Procedures, Wiley-Interscience, a Division of John Wiley and Sons, Inc., New York, N.Y., 1971.

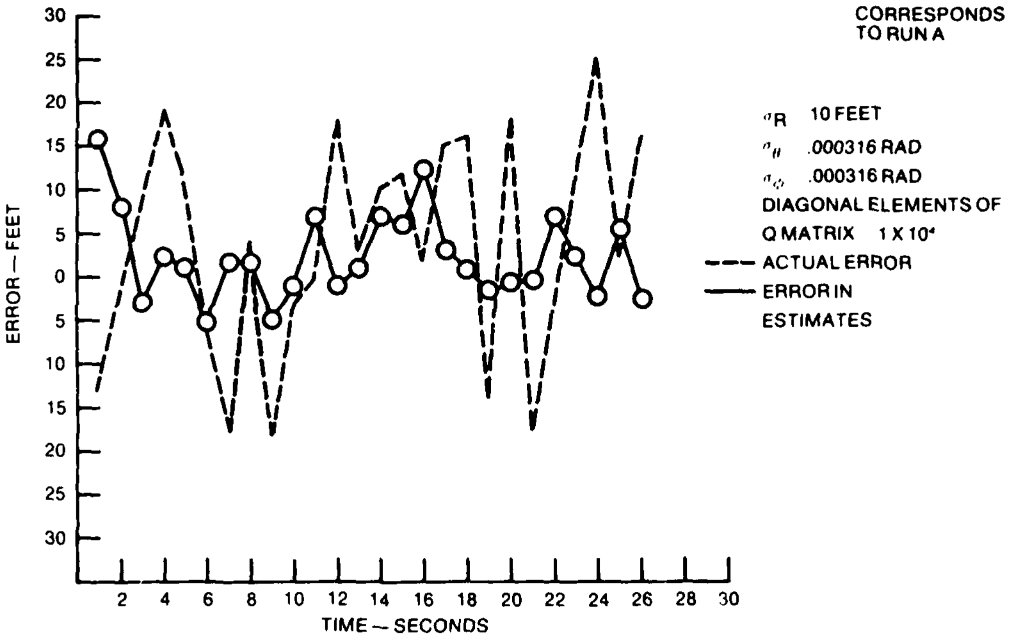

Figure 8. Actual measurement error and error of estimated state (X component) vs time.

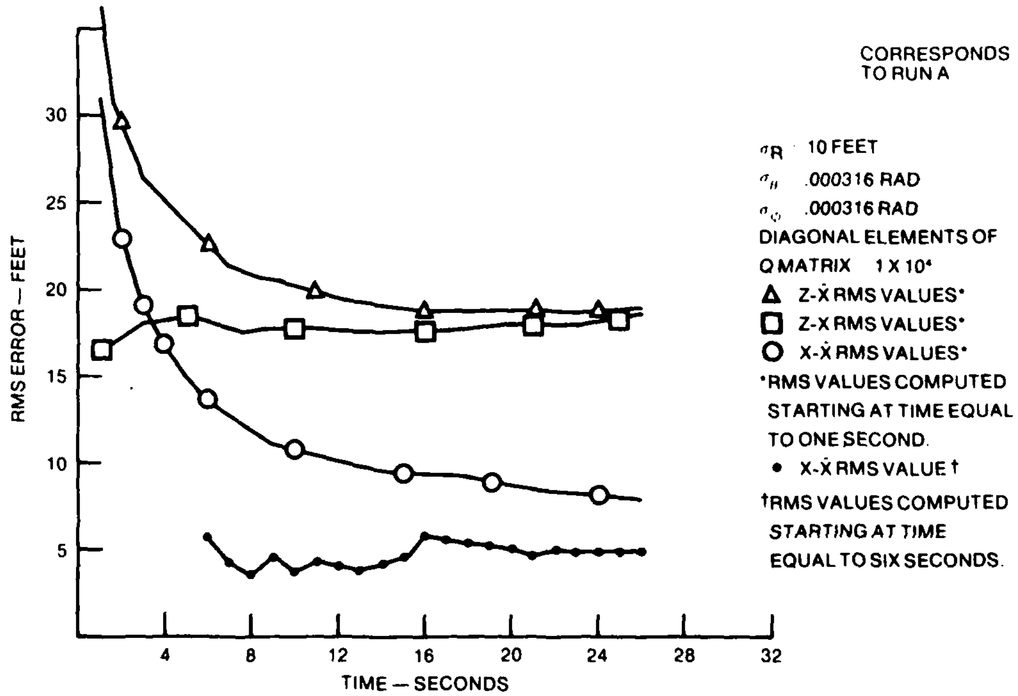

Figure 9. RMS errors for extended Kalman filter (X component) vs time.

Figures 10 and 11 illustrate RMS error curves for the low noise case where the diagonal elements of Q have been varied. In figure 10 the diagonal elements are 1×10^5. In figure 11 the diagonal elements of the Q matrix are 1×10^4. It is quite evident that the Q matrix plays an important role in estimating the elements of the state vector. It was noted that the convergence rate of the estimator was higher for higher values of the Q matrix. This did not necessarily mean better estimates. It was found that there was an optimum value for the diagonal elements of the Q matrix which yielded "best estimates."

Figures 12 and 13 are the data for Case 1, the low noise case, of the y component of the position element of the state vector. An interesting aspect is noted from this data--the geometry has a large impact on the noise level of the y component of the sensor data. Again, the filter does a good job of obtaining estimates of the y element of the state vector. The ratio is approximately 4 to 1.

Figures 14 through 16 present data on Case 2 (the high level noise case) for the x component of the state vector. For comparison, compare figure 14 to figure 8 and figure 15 to figure 9. It is quite obvious that the filter does an excellent job in obtaining estimates of the position elements of the state vector. It can be seen by studying figure 14 that actual errors can be as large as 450 feet while the estimates are under 80 feet. The RMS data of figure 15 is quite interesting. The RMS values were calculated using equation 6.1 for data after $t = 6$ seconds. That is, the data before time reached 6 seconds was not included in the RMS calculations. The estimates, for the data of figure 15 were three to four times better than the actual raw measurements. The values of the elements along the diagonal of the Q matrix were varied from 1×10^4 for the data of figure 15 to 1×10^5 for the data of figure 16. The estimates were closer to the actual values of the state vector (x component) by a factor of approximately 1.25. Thus the same phenomenon is noted for the high noise case as for the low noise case--the estimates can be made closer to the actual values by weighting the covariance matrix.

6.2.2 Linear Kalman Filter Analysis Results

Figures 17, 18 and 19 present data on the performance of the decoupled linear Kalman filter where the noise variance of the observation data was transformed via equations (87) from spherical to Cartesian reference frames. Figure 17 presents the curves for the error in the actual data and the error in the estimated value of the state. The decoupled linear Kalman filter did not perform very well as illustrated in figure 17. The error in the estimated state was as large or larger than the error in the observation data. The RMS values of the errors ($x - \hat{x}$, and $z - x$) are presented in the curves of figure 18. The figure illustrates that the estimates were poorer than the raw data. Again, the diagonal elements of the Q matrix were varied. It can be seen from the data of figure 18 that the filter performance can be improved by varying the elements of the Q matrix.

Figure 19 presents the RMS error data for the y component of the position element of the state vector. Again, it is noted, as was the case of the extended Kalman filter, that the error in the observation data is a function of the geometry. The error increases as the y component of the state vector increases.

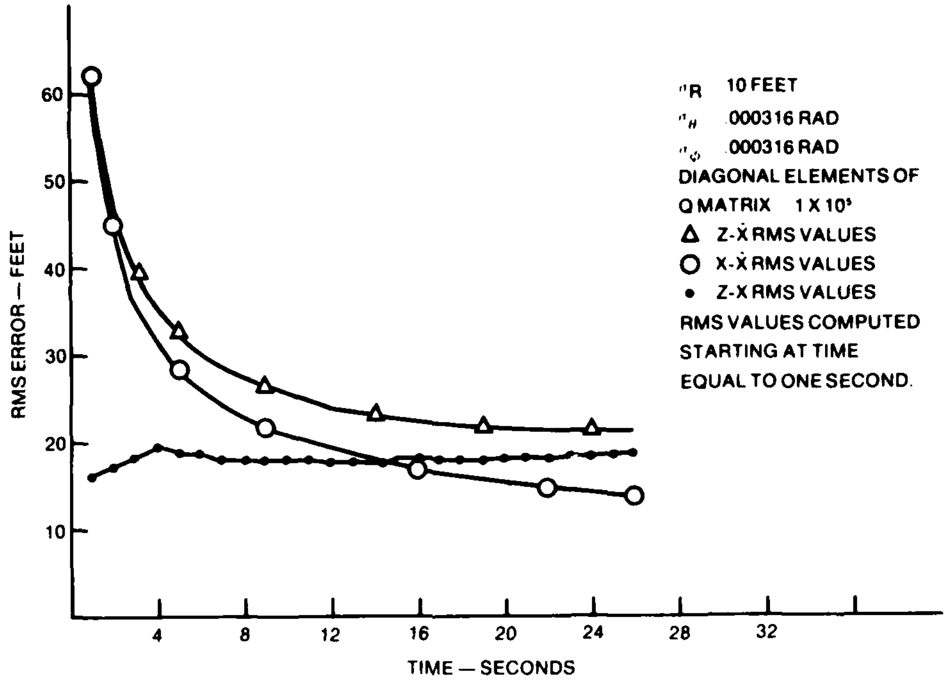

Figure 10. RMS errors for extended Kalman filter (X component) vs time.

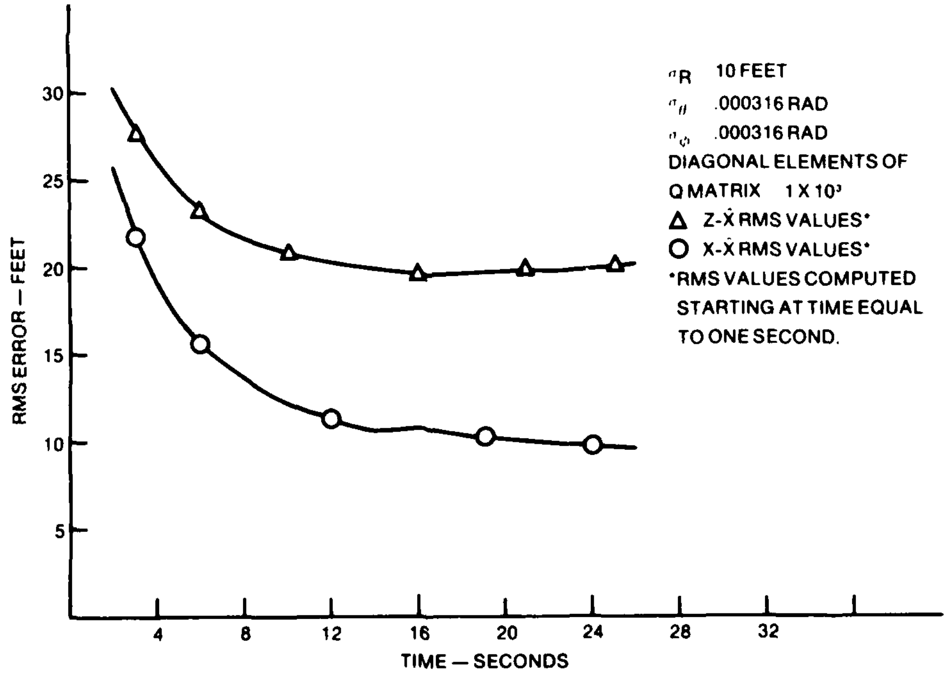

Figure 11. RMS error for extended Kalman filter (X component) vs time.

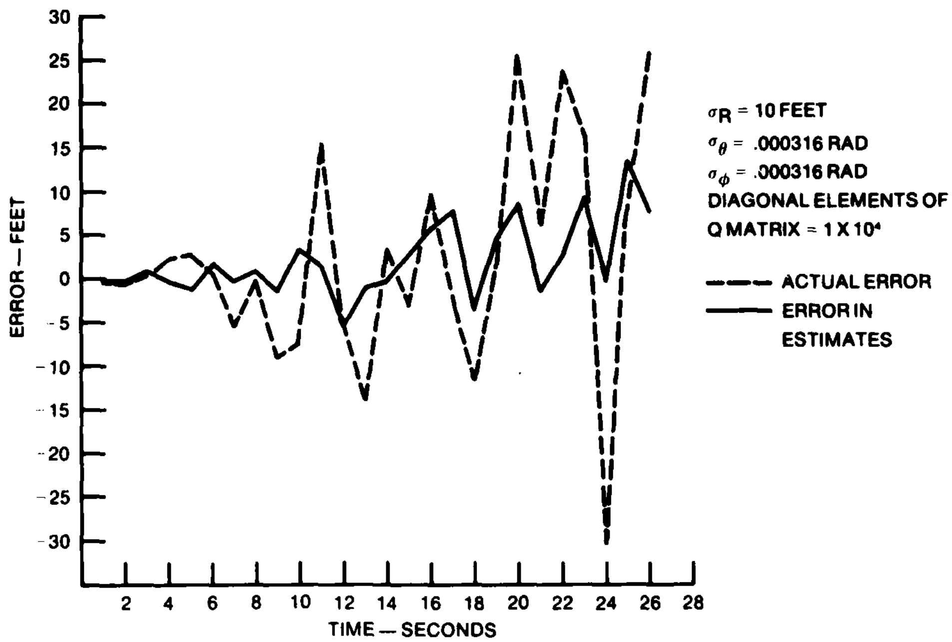

Figure 12. Actual measurement error and error of estimated state (Y component) vs time.

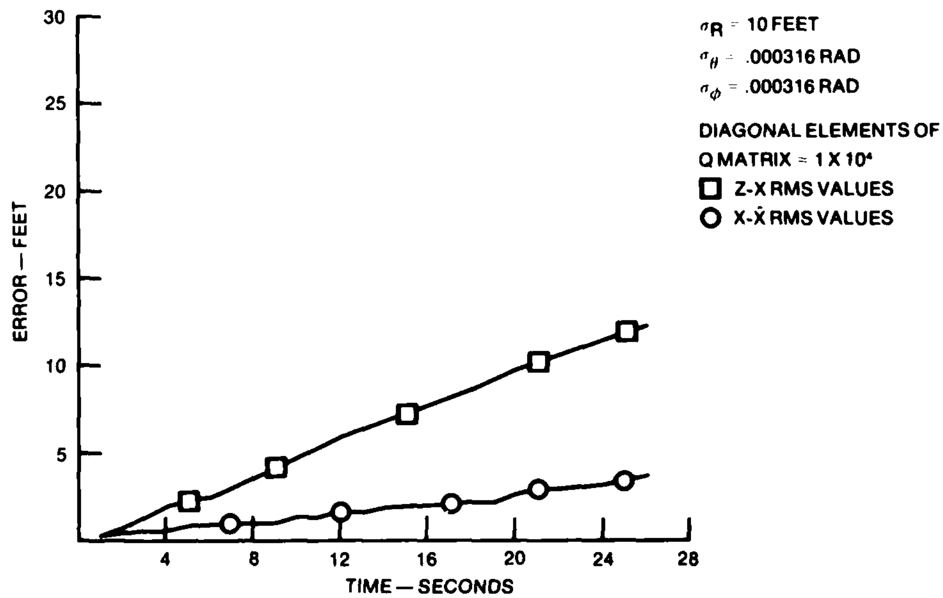

Figure 13. RMS error for extended Kalman filter (Y component) vs time.

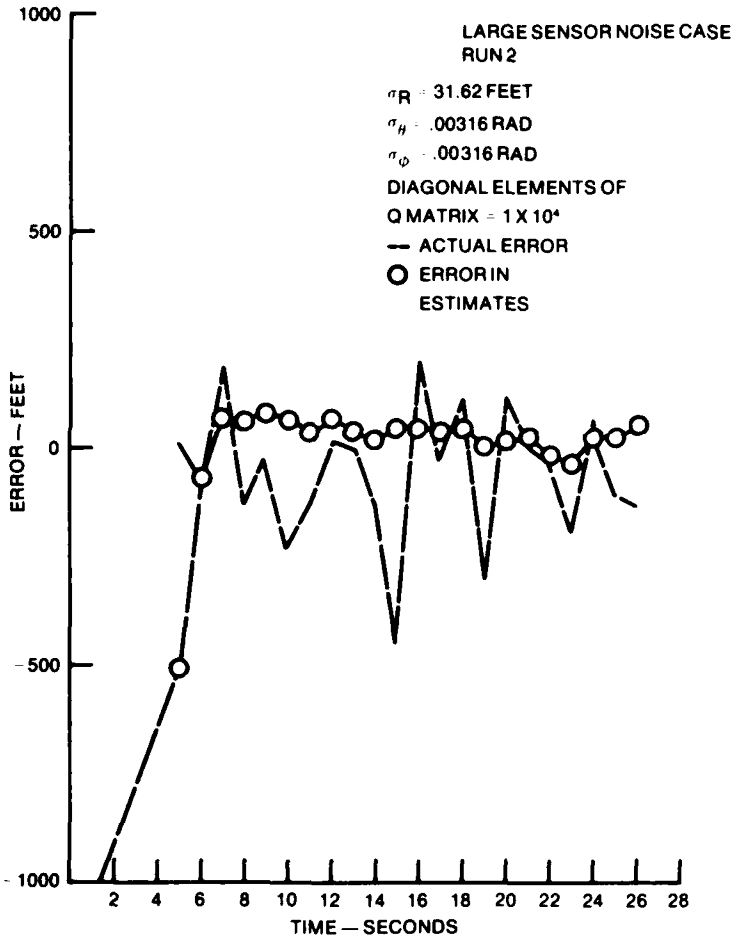

Figure 14. Actual measurement of error and error of estimated state (X component) vs time.

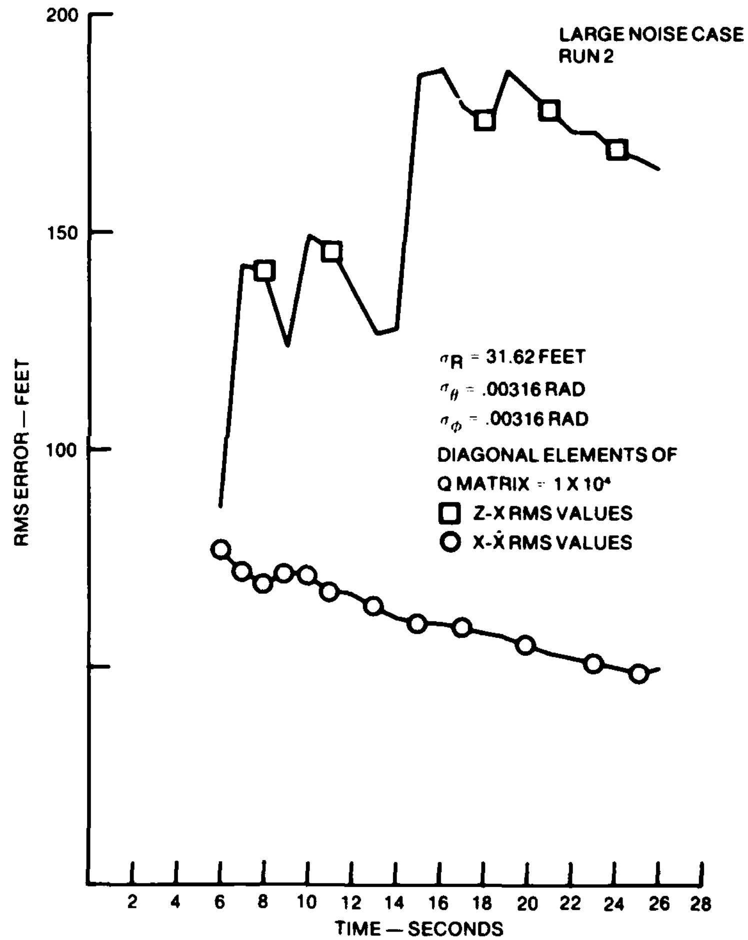

Figure 15. RMS errors for extended Kalman filter (X component) vs time.

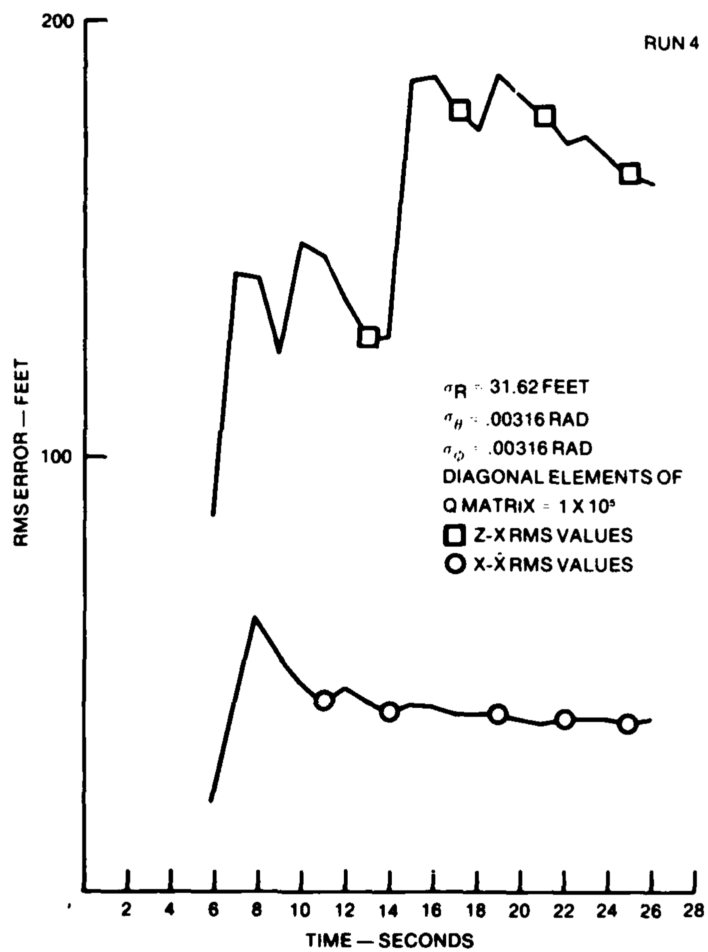

Figure 16. RMS errors for extended Kalman filter (X component) vs time.

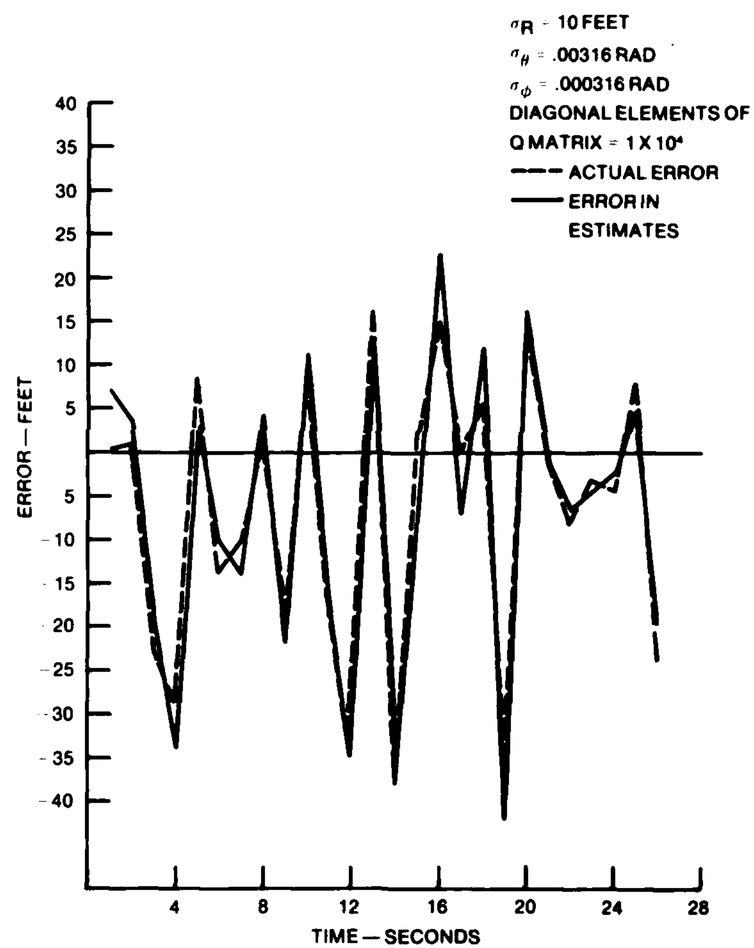

Figure 17. Actual measurement of error and error of estimated state (X component/linear filter) vs time.

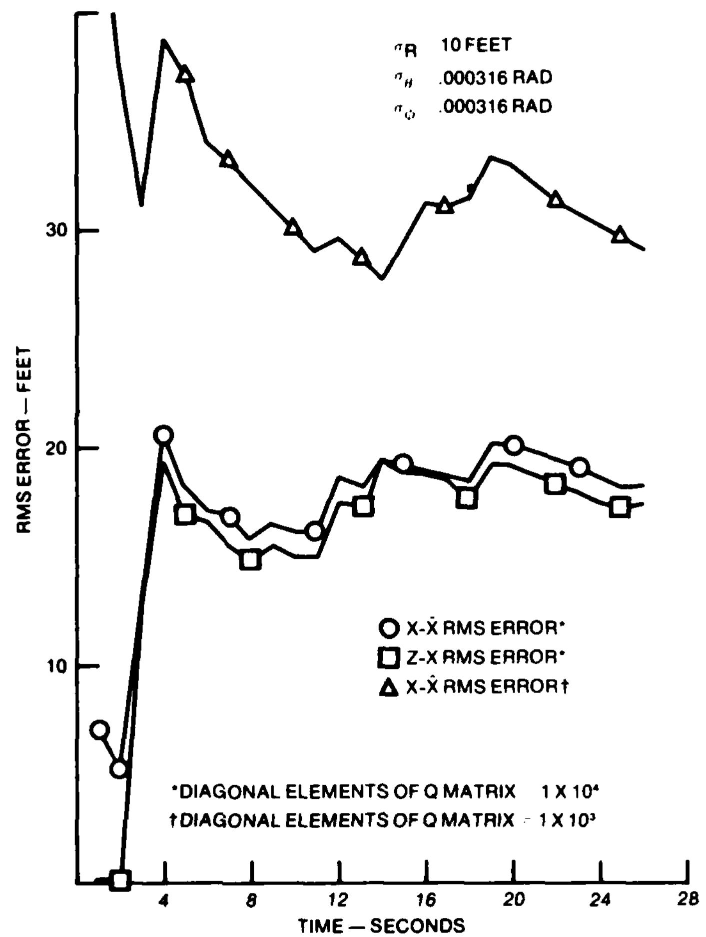

Figure 18. RMS error for linear Kalman filter (X component) vs time.

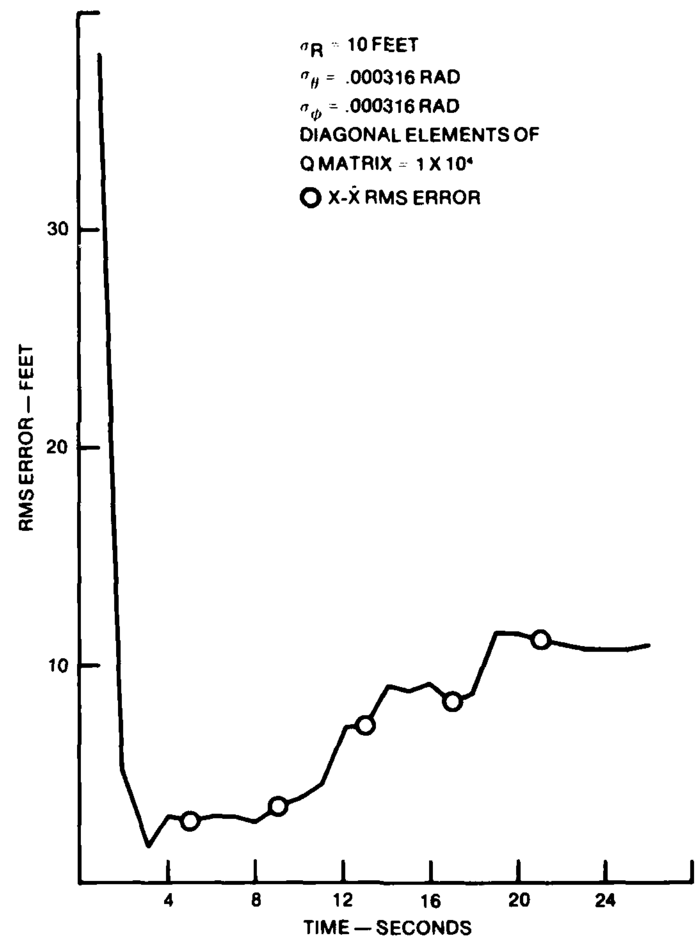

Figure 19. RMS error for linear Kalman filter (Y component) vs time.

7. CONCLUSIONS AND RECOMMENDATIONS

The validation of missile system performance specifications from test data (captive and free-flight) requires a high level of confidence in the observation data recorded during the test. This study was directed at raising the confidence level of that test data.

The methodology of approaching this problem is outlined in figure 4 which illustrates two areas of effort as delineated by the two differently shaded areas: (1) completed effort and (2) recommended follow-on. The initial effort conducted was the development of a sequential tracking algorithm for two different approaches, the validation of the tracking algorithm using a 6-degree of freedom missile simulation to supply the simulated measurements of the missile in flight, and the recommendation of the preferred approach based on comparative analysis.

The two different algorithms that were implemented and validated were (1) the decoupled linear Kalman filter and (2) the extended Kalman filter (nonlinear measurements).

7.1 CONCLUSIONS

The performance of the extended Kalman filter was superior to that of the linear filter approach, hence the recommended approach is the extended Kalman filter working in a Cartesian reference frame with nonlinear spherical coordinate frame observation data. The extended Kalman filter worked very well with high- and low-level noise observation data. The estimates were reduced three to five times (on an RMS ratio basis) than the raw measurement data. The recommended filter's performance was a function of a weighting of the covariance matrix. This weighting affected the convergence rate and the level of the estimates at filter turn-on and during the period just following filter turn-on. The R matrix for the extended Kalman filter was constant as opposed to the R matrix which varied as a function of time for the linear filter approach.

The use of an extended Kalman filter as defined in this report greatly enhances the level of confidence in defining the missile's free-flight trajectory--raw data errors (RMS) on the order of 20 feet are reduced to RMS errors of 5 feet or less!

7.2 RECOMMENDATIONS

It is recommended that a system engineering effort be conducted as outlined by the shaded areas identified as "follow-on" in figure 4. This effort would be in the following areas:

(1) Adaptive Kalman filtering

(a) Maneuver gates

(b) Weighted gain matrix

(2) Initialization

(a) Timing

(b) Initial values of covariance, state vector, and measurement sensor noise

(3) Terminal intercept performance

(a) Variance of line-of-sight rates measured by guidance sensors

These three areas would be the primary areas of research and analysis. Secondary areas of investigation would be:

(4) Nonlinear dynamics

(5) Smoothing of optimal estimates, and

(6) Multiple sensor observations.

Some interesting observations were made during the first study phase at the Naval Ocean Systems Center, relative to adaptive Kalman filters. It was noted that by varying the covariance matrix during the initialization phase of the trajectory that the rate of conveyance of the estimated values of the state vector could be controlled. This was an open loop form of control. One of the adverse effects was that the filter could become unstable. A methodology was investigated, but not implemented, that would incorporate a form of closed loop control on the initial variation of the covariance matrix. This in effect would be an adaptive filter. The gain matrix would then be weighted as a function of the variations in the covariance matrix. An algorithm would be developed, if appropriate, to effect this closed loop control of the Kalman filter convergence. Other schemes to be investigated in formulating the adaptive filter would be to monitor the residuals to determine if a large maneuver had occurred. The observation data would then be weighted relative to the estimated maneuver before being processed by the Kalman filter.

Initialization of the filter is another area of recommended research. Methodology for determining when to start the filter and how to estimate the initial values of the covariance matrix and initial values of the state vector should be investigated.

The final area of recommended research is to analyze the terminal intercept performance. Techniques for determining accurate values of the line-of-sight rates as observed by the guidance sensor during the final 3000 feet of flight trajectory should be analyzed. The problem associated with determining accurate line-of-sight rates (LOSR) from observation data of the reconstructed missile and target flight trajectories is that the LOSR is a function of one over the square of relative range betwee.ı the missile and target. As this range goes to zero the LOSR goes to infinity. When there is

a large uncertainty in the relative range the variance of the LOSR becomes large. This is the reason why the estimates of the trajectories need to be optimized with the least error variances. The reason for determining the LOSR is to compare it to measured line-of-sight rates as observed by the guidance sensor for performance evaluation. If the methodology for determining the actual line-of-sight rates has a higher error variance than the variance of the guidance sensor, it makes little sense to do a performance evaluation. The standard must have a higher level of confidence associated with it as compared to the item being compared against the standard. A possible solution to this problem would be to utilize the inertial reference system on board the missile itself. Alignment errors would be estimated during the initial and midcourse phases of the flight trajectory; therefore, during the terminal intercept phase a higher degree of confidence in the missile's trajectory could be realized.

Secondary areas of recommended research that should be investigated deal with optimal smoothing, nonlinear dynamics and multiple sensors to collect the observation data. A brief outline of these areas of research follows.

The initial study performed at the Naval Ocean Systems Center utilized linear dynamics with nonlinear measurements in the extended Kalman filter model. The assumption was made that the acceleration data was available at a high data rate and therefore constant over the sampling interval. The transformations of the acceleration from missile body to inertial reference frames were done outside the filter. If a reference frame were utilized where the system dynamics was a nonlinear combination of the states the extended Kalman filter would have to be modified to incorporate the nonlinear system dynamics.

Optimal smoothing of data is another means of processing raw data. This smoothing process can also be applied to data that has been processed by a sequential estimation filter. This area of research would investigate alternate smoothing techniques for processing the observation data.

Multiple sensor observation is an extension of the single sensor problem. The data from the ground-based sensors positioned at various locations throughout the test range would each generate an estimate of the missile's and target's trajectory. These trajectories could then be processed through a smoother to arrive at a finalized version of the trajectories.

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APPENDIX A

DEFINITIONS OF COMPUTER PROGRAMS

APPENDIX A

DEFINITIONS OF COMPUTER PROGRAMS

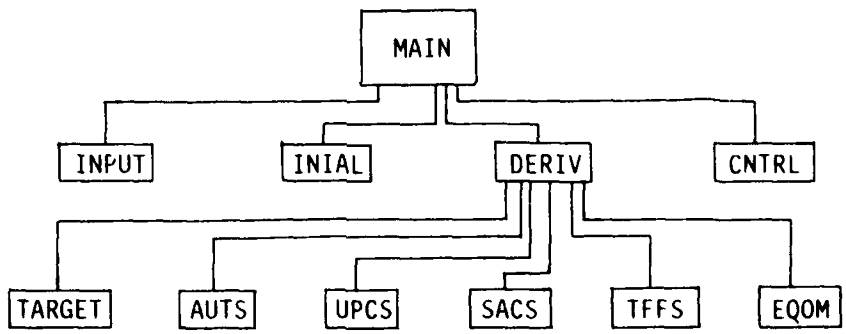
A.1 OVERVIEW OF 6-DOF TRAJECTORY SIMULATION

A.1.1 General Program Flow

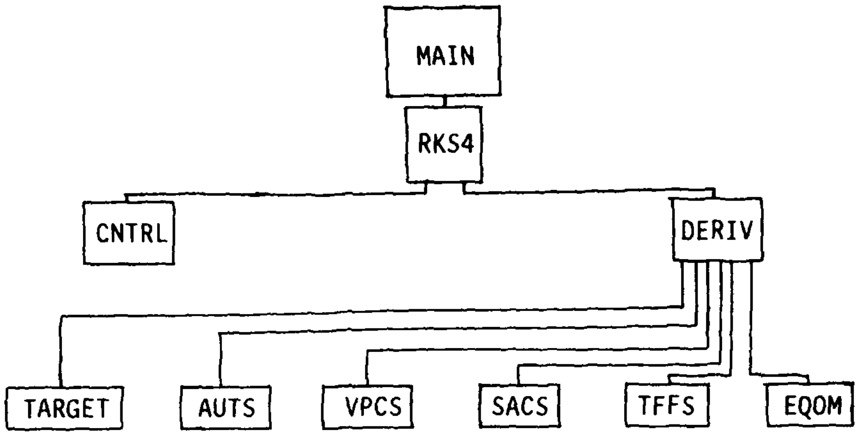
The program is broken into seven stages, indicated by variable IENTRY, as follows:

IENTRY	STAGE DESCRIPTION
1	Pre-data initialization
2	Post-data initialization
3	Trajectory computation
4	Not currently used
5	Not currently used
6	Periodic print out
7	Post-flight computations, print and plot

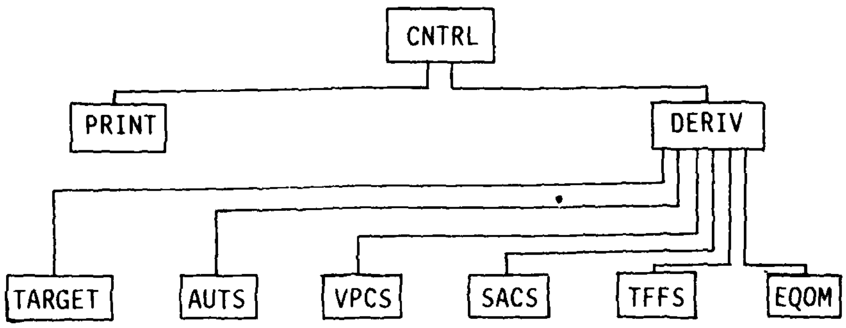
The program routines involved in each stage are shown in the following diagrams.



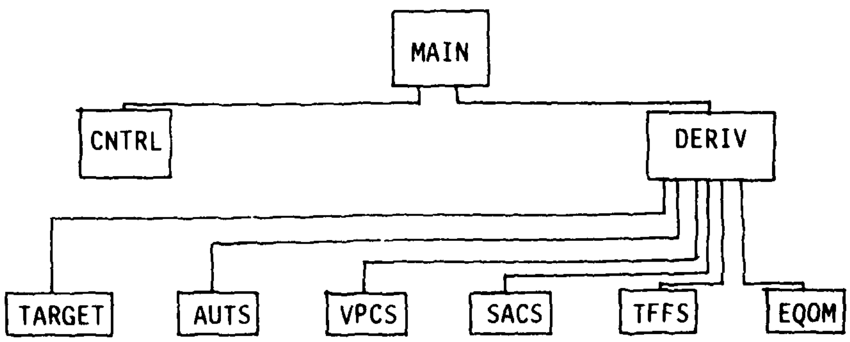
IENTRY = 1 & 2



IENTRY = 3



IENTRY = 6



IENTRY = 7

A.1.2. Subroutine Descriptions

MAIN -- Allocates storage, defines error tolerances, and sets up initial indices for the integration routine.

INPUT -- Reads and stores input data

INIAL -- Computes initial parameters from inputs

BLK1 -- BLOCK DATA--Constants and reference values

BLK2 -- BLOCK DATA--Thrust tables

BLK3 -- BLOCK DATA--Aerodynamic coefficients

DERIV -- Calls routines to compute differential equations

CNTRL -- Controls program flow as to breakup times, print intervals and program stops.

TARGET -- Computes target trajectory

RELT -- Relative missile-to-target kinematics

AUTS -- Model of control system (auto pilot)

SACS -- Aerodynamic forces and moments

GUIDE -- Model of missile guidance scheme

FCC -- Fire control computer logic

VPCS -- Vehicle physical characteristics

TFFS -- Computes thrust components

RKS4 -- Numerical integration routine--Fourth order Runge-Kutta with Simpsons Rule check--Variable or fixed interval.

EQOM -- Equations of motion

PRINT -- Basic trajectory print out

ATMOS -- Computes atmospheric pressure and velocity of sound from IACO 1962 model atmosphere.

INVR, M3X31, TMULT -- Matrix/vector manipulation

LOOK1, LOOK2, LOOK3 -- Lookup routines for tabular data

GAUSS, RANDU -- Noise generator

A.2 KALMAN FILTER ROUTINE DESCRIPTIONS

A.2.1 General Description and Use of Kalman Filter in 6-DOF

A tracking type Kalman filter was inserted into the trajectory simulation to test the filter's accuracy. Uncluttered trajectory information was taken from the simulation, Gaussian noise was added and the results were fed into the filter. The filter had no effect on the trajectory.

The software for the Kalman filter consisted of the following routines:

KALDRV -- driver routine to set up observation and forcing function.

KALFT -- Kalman filter computations

ACXFM -- adds noise to the missile body x, y, z accelerations then transforms them to inertial coordinates.

MMULT -- performs matrix multiplications

DINVER -- performs matrix inversion

MATIO -- performs matrix input/output

The filter is connected to the simulation through the CNTRL routine which calls KALDRV. Logic was added to the CNTRL routine to call KALDRV in stage 2 and then to call KALDRV in stage 3 at time zero and thereafter every DTKAL seconds. DTKAL is an input variable which indicates the Kalman filter time interval. CNTRL also tells KALDRV to print filter information at the same interval as the simulation.

A.2.2 Kalman Filter Routine Documentation (Extended Kalman Filter)

A. KALDRV

1. USAGE: Call KALDRV (TIME, IKALP, DTKAL)

2. PURPOSE: To initiate and drive the Kalman filter.

3. SUBROUTINES REQUIRED:

   MATIO -- matrix input/output routine

   MMULT -- matrix multiplication routine

   GAUSS -- Gaussian distribution random number generator

   ACXFM -- routine to set up the forcing function

   KALFT -- the Kalman filter calculations

4. ARGUMENTS:

Input: TIME -- trajectory simulation time, in seconds

IKALP -- print flag, IKALP = 0 → output filter data, IKALP = 1 → call filter but perform no output

Output: DTKAL -- Kalman filter time interval, in seconds

5. COMMON VARIABLES: all variables in COMMON are inputs.

COMMON AREA ENT

IENTRY -- indicates the stage of the trajectory simulation

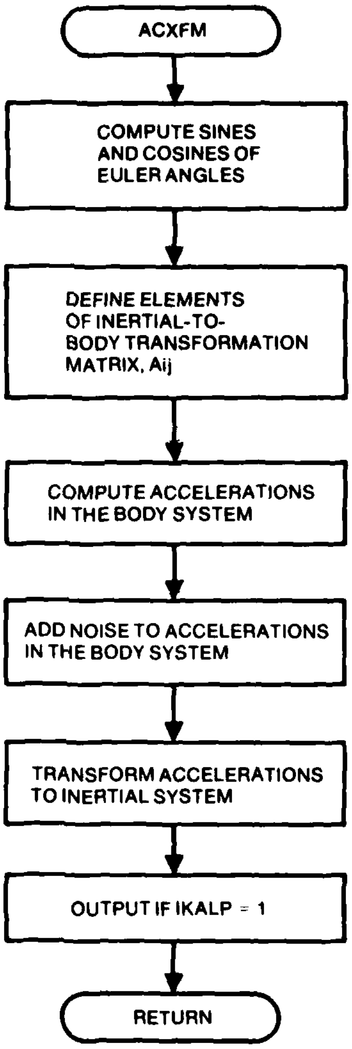
COMMON AREA DER

DY -- array of simulation derivatives

Y -- array containing the results of integrating DY

Note that only three variables from each Y and DY are used, namely Y(4), Y(5), Y(6) and DY(4), DY(5), DY(6). These are equivalenced to XG, YG, ZG and XGDOT, YGDOT, ZGDOT, respectively, and are the missile position and velocity vectors in the inertial (ground) coordinate system.

6. FLOWCHART:


ACXFM
COMPUTE SINES AND COSINES OF EULER ANGLES
DEFINE ELEMENTS OF INERTIAL-TO-BODY TRANSFORMATION MATRIX, Aij
COMPUTE ACCELERATIONS IN THE BODY SYSTEM
ADD NOISE TO ACCELERATIONS IN THE BODY SYSTEM
TRANSFORM ACCELERATIONS TO INERTIAL SYSTEM
OUTPUT IF IKALP = 1
RETURN

7. OUTPUTS: all inputs are echoed on output

R, THETA, PHI -- position of missile in spherical coordinates, without noise

z -- observation vector, three element vector containing the missile position, in spherical coordinates.

FF -- forcing function vector, three element vector containing the missile x, y, z accelerations, in the inertial coordinate system, with noise

x -- estimated state vector, six element vector containing the estimated state from the Kalman filter, x(1), x(2), x(3) are the estimated coordinates of position and x(4), x(5), x(6) are the estimated velocities, both in rectangular coordinates

OBXYZ -- three element vector containing the observation in rectangular coordinates

zMx -- six element vector containing the actual value of each state, from the simulation, minus the estimated value from the Kalman filter

RMS -- six element vector containing the accumulated RMS error (actual minus estimated)

OBXMXG -- three element vector containing the actual value of position, from the simulation, minus the observed value (actual plus noise), in spherical coordinates

OBXMX -- three element vector containing the observed values of position minus the estimated values (i.e., OBXYZ(i) - X(i), i = 1, 2, 3)

RMS1 -- three element vector containing the accumulated RMS error of position (actual minus observed)

RMS2 -- three element vector containing the accumulated RMS error of position (observed minus estimated)

PK -- the current value of the error covariance matrix associated with the current estimate of the state vector

G -- the current value of the Kalman gain matrix

B. ACXFM

1. USAGE: CALL ACXFM (IX, IKALP, VAX, VAY, VAZ, AX, AY, AZ)

2. PURPOSE: To calculate the missile accelerations, with noise, in the inertial coordinate system.

3. SUBROUTINES REQUIRED:

   GAUSS -- Gaussian distribution random number generator

4. ARGUMENTS:

   Input: IX -- seed for GAUSS

   IKALP -- print flag, IKALP = 1 → print, IKALP = 0 → no print

   VAX, VAY, VAZ -- standard deviations for X, Y and Z acceleration noises

   Output: AX, AY, AZ -- X, Y and Z components of acceleration in the inertial coordinate system with noise.

5. COMMON VARIABLES: all variables in COMMON are input

   COMMON AREA PPP

   THETAP, PSIP, PHIP -- missile Euler angles

   GX, GY, GZ -- X, Y and Z components of gravity in the body system

   COMMON AREA DER

   DY -- array of simulation derivatives

   Y -- array containing the results of integrating DY. The elements of Y and DY that are needed are equivalenced to meaningful variables names as follows.

   Y(1) ≡ U, Y(2) ≡ V, Y(3) ≡ W -- these are the missile X, Y and Z velocities in the body system.

   Y(7) ≡ PP, Y(8) ≡ QQ, Y(9) ≡ RR -- these are the roll, pitch and yaw angular rates in the body system.

   DY(1) ≡ UDOT, DY(2) ≡ VDOT, DY(3) ≡ WDOT -- these are the missile X, Y and Z accelerations in the body system.

5. FLOWCHART:

GIVEN: Φ — THE TRANSITION MATRIX OF THE SYSTEM

Q — COVARIANCE MATRIX OF RANDOM FORCING FUNCTION NOISE

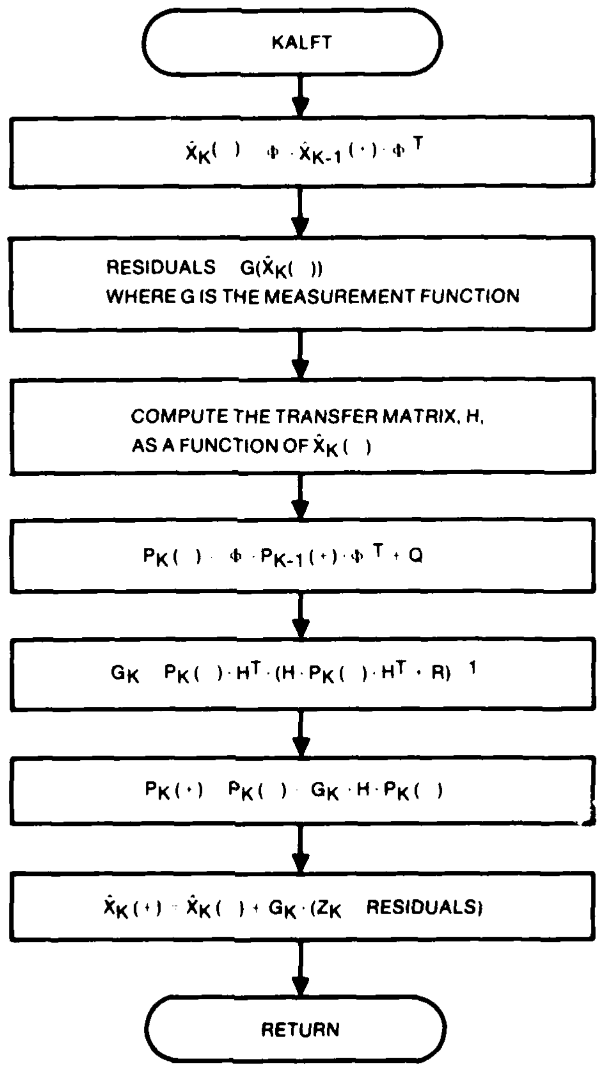
R — COVARIANCE MATRIX OF RANDOM OBSERVATION NOISE

ff_{K-1} — FORCING FUNCTION VECTOR OF PREVIOUS ITERATION

$\hat{X}_{K-1}(+)$ — PREVIOUS ESTIMATE OF STATE VECTOR

$P_{K-1}(+)$ — ERROR COVARIANCE MATRIX OF PREVIOUS ESTIMATE

$\vec{Z}_K$ — CURRENT OBSERVATION



C. KALFT

1. USAGE: CALL KALFT (IKALP, PHI, FF, Q, R, Z, P, G, X)

2. PURPOSE: To perform one iteration of the tracking type Kalman filter with a forcing function.

3. SUBROUTINES REQUIRED:

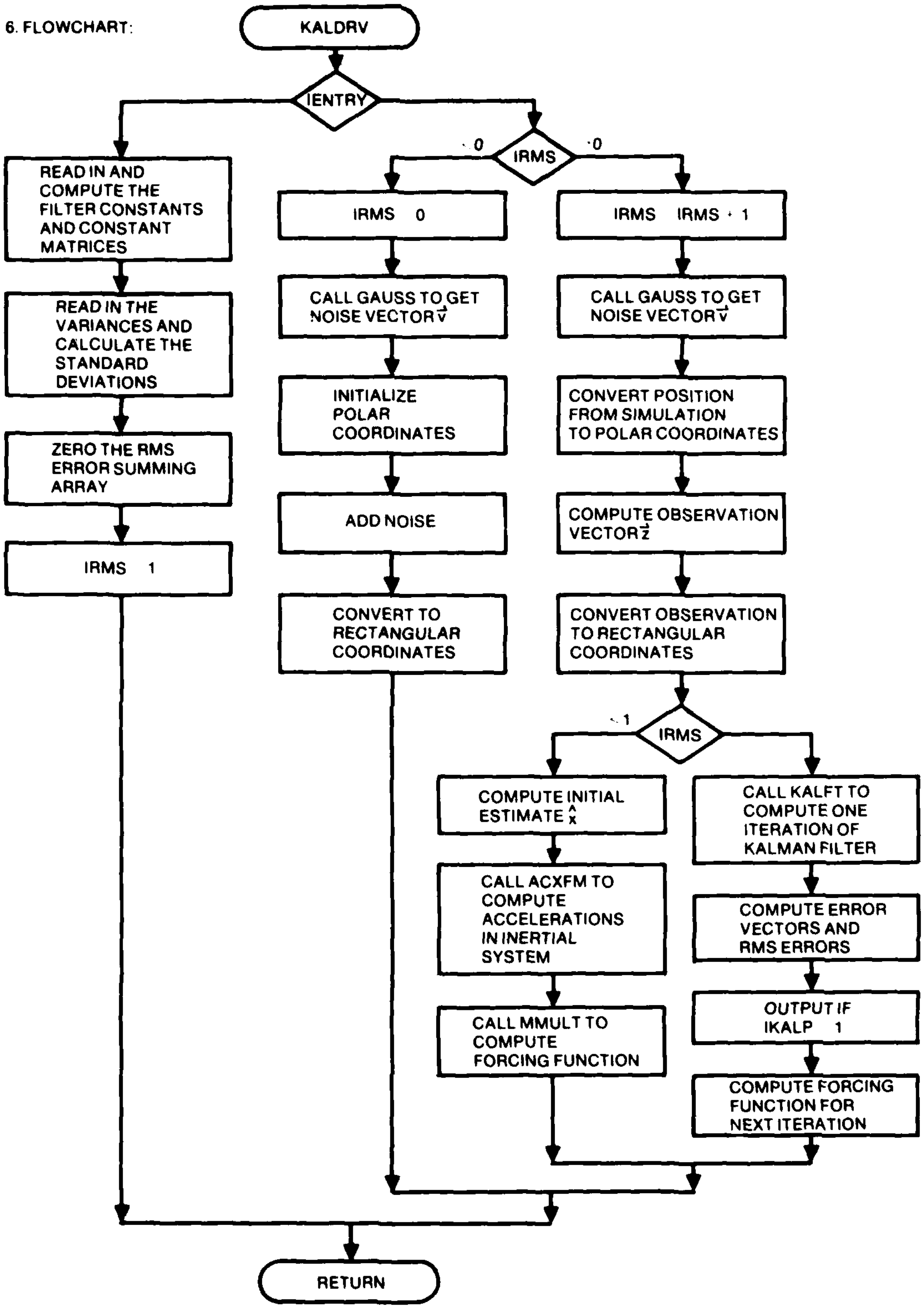
MMULT -- matrix multiplication routine

DINVER -- double precision matrix inversion routine

4. ARGUMENTS:

Input:	IKALP	-- print flag, IKALP = 1 → print, IKALP = 0 → no print
	PHI	-- transition matrix of the system (6x6)
	FF	-- forcing function vector (3)
	Q	-- covariance matrix of forcing function statistics (6x6)
	R	-- variance matrix of observation noise (3x3)
	Z	-- observation vector (3)
Output:	P	-- error covariance matrix associated with the previous estimate of the state vector (6x6)
	G	-- Kalman gain matrix (6x3)
	X	-- estimated state vector (6)

6. FLOWCHART:


KALDRV
IENTRY
READ IN AND COMPUTE THE FILTER CONSTANTS AND CONSTANT MATRICES
READ IN THE VARIANCES AND CALCULATE THE STANDARD DEVIATIONS
ZERO THE RMS ERROR SUMMING ARRAY
IRMS 1
IRMS
0
0
IRMS 0
CALL GAUSS TO GET NOISE VECTOR $\vec{v}$
INITIALIZE POLAR COORDINATES
ADD NOISE
CONVERT TO RECTANGULAR COORDINATES
IRMS IRMS + 1
CALL GAUSS TO GET NOISE VECTOR $\vec{v}$
CONVERT POSITION FROM SIMULATION TO POLAR COORDINATES
COMPUTE OBSERVATION VECTOR $\vec{z}$
CONVERT OBSERVATION TO RECTANGULAR COORDINATES
IRMS
1
COMPUTE INITIAL ESTIMATE $\hat{x}$
CALL ACXFM TO COMPUTE ACCELERATIONS IN INERTIAL SYSTEM
CALL MMULT TO COMPUTE FORCING FUNCTION
CALL KALFT TO COMPUTE ONE ITERATION OF KALMAN FILTER
COMPUTE ERROR VECTORS AND RMS ERRORS
OUTPUT IF IKALP 1
COMPUTE FORCING FUNCTION FOR NEXT ITERATION
RETURN

APPENDIX B

COORDINATE TRANSFORMATIONS

APPENDIX B

COORDINATE TRANSFORMATIONS

All coordinate transformations used in this simulation are composed of standard angles and rotations. The order of the rotations is always yaw, pitch and roll.

Inertial Axes to Missile Body Axes: The relationship between the missile body and inertial axes systems is described by the yaw, pitch and roll angles, as shown in Figure B.1. The transformation is given by the following matrix relation (see ref. 9):

$$\begin{bmatrix} \bar{i}_B \\ \bar{j}_B \\ \bar{k}_B \end{bmatrix} = \begin{bmatrix} \cos A\cos\Psi & \cos A\sin\Psi & -\sin A \\ \sin\Gamma\sin A\cos\Psi - \cos\Gamma\sin\Psi & \sin\Gamma\sin A\sin\Psi + \cos\Gamma\cos\Psi & \sin\Gamma\cos A \\ \cos\Gamma\sin A\cos\Psi + \sin\Gamma\sin\Psi & \cos\Gamma\sin A\sin\Psi - \sin\Gamma\cos\Psi & \cos\Gamma\cos A \end{bmatrix} \begin{bmatrix} \bar{i}_I \\ \bar{j}_I \\ \bar{k}_I \end{bmatrix} \tag{B.1}$$

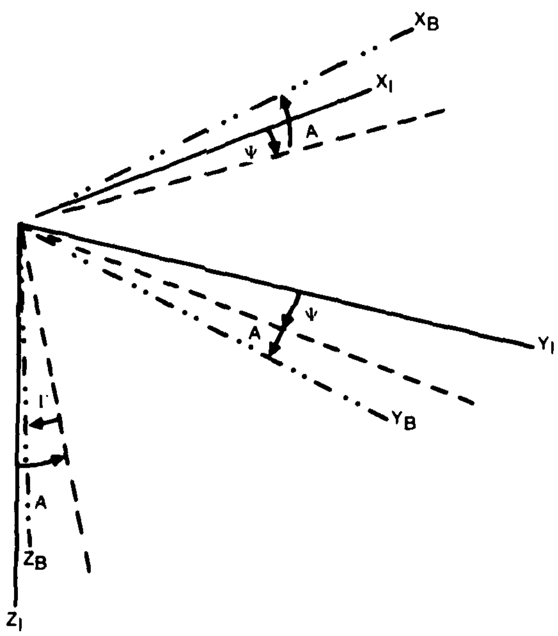


Figure B.1. Inertial axes to missile body axes.

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